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ON THE NON-$(p-1)$-PARTITE K_p-FREE GRAPHS

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Abstract

We say that a graph G is *maximal* K_p*-free* if G does not contain K_p but if we add any new edge $e \in E(\overline{G})$ to G, then the graph $G + e$ contains K_p. We study the minimum and maximum size of non-$(p-1)$-partite maximal K_p-free graphs with n vertices. We also answer the interpolation question:

for which values of n and m are there any n-vertex maximal K_p-free graphs of size m?

Keywords: extremal problems, maximal K_p-free graphs, K_p-saturated graphs.

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1. Introduction and Notation

We consider finite undirected graphs without loops or multiple edges. A graph G has a vertex set $V(G)$ and an edge set $E(G)$. The *size* of a graph is the number of edges. We denote by $e(G)$ the size of the graph G and by $v(G)$ the number of vertices of G. The set of neighbours of a vertex $v \in V(G)$ is denoted by $N_G(v)$, or briefly by $N(v)$. Moreover, $N_G[v] = N_G(v) \cup \{v\}$. Let $S \subseteq V(G)$, $N_G[S] = \bigcup_{v \in S} N_G[v]$. By $G[S]$ we denote the subgraph induced by the set of vertices S. The degree of v is denoted by $d_G(v)$. If H is a subgraph of G and $v \in V(G)$, then $d_H(v) = |N_G(v) \cap V(H)|$. For $S \subseteq V(G)$ we write $d_S(v) = d_{G[S]}(v)$. We also use the following notation: S_n is the star with n vertices, K_n is the complete graph with n vertices, for $k \geq 2$, $K_{n_1,\ldots,n_k}$ is the complete k-partite graph.

For undefined concepts we refer the reader to [4].

Let n, p be integers and $p \geq 2$. We say that the graph G is *K_p-free* if G does not contain K_p as a subgraph. We say that G is *maximal K_p-free* (sometimes called *K_p-saturated*) if G does not contain K_p as a subgraph but if we add any new edge $e \in E(\overline{G})$ to G, then the graph $G + e$ contains K_p. The set of all maximal K_p-free graphs of order n is denoted by $\mathrm{M}(n, K_p)$. A complete k-partite graph $K_{n_1,\ldots,n_k}$ such that $|n_i - n_j| \leq 1$ for $i, j = 1, \ldots, k$ and $n_1 + \cdots + n_k = n$ we call *Turán's graph* and denoted $T_k(n)$. The classical theorem of Turán [12] states that if G is an n-vertex K_p-free graph of maximum size, then G is isomorphic to $T_{p-1}(n)$. On the other hand Erdős, Hajnal and Moon [5] proved that if G is maximal K_p-free with $n \geq p - 1$ vertices, then $e(G) \geq (p-2)n - \frac{1}{2}(p-1)(p-2)$. However, every maximal K_p-free graph from this theorem is $(p-1)$-partite and contains a vertex of degree $n - 1$. The problem of determining the minimum size of maximal K_p-free graphs with no vertex of degree $n - 1$ was studied by Alon *et al.* [1]. The case for $p = 3$ was treated by Füredi, Seress [8] and Erdős, Holzman [6]. Duffus and Hanson [7] study the minimum size of maximal K_p-free graphs with fixed minimum degree.

We will consider the maximal K_p-free graphs that are not $(p-1)$-partite. Let $\mathrm{s}(n, K_p)$ and $\mathrm{e}(n, K_p)$ denote minimum and maximum size of a maximal K_p-free graph with n vertices that is not a $(p-1)$-partite graph, i.e.,

$$\mathrm{s}(n, K_p) = \min\{e(G) : G \in \mathrm{M}(n, K_p) \text{ and } G \text{ is non-}(p-1)\text{-partite}\},$$
$$\mathrm{e}(n, K_p) = \max\{e(G) : G \in \mathrm{M}(n, K_p) \text{ and } G \text{ is non-}(p-1)\text{-partite}\}.$$

Let us define the following sets of graphs:

$$\mathrm{S}(n, K_p) = \{G \in \mathrm{M}(n, K_p)\colon\ e(G) = \mathrm{s}(n, K_p) \text{ and } G \text{ is non-}(p-1)\text{-partite}\},$$
$$\mathrm{E}(n, K_p) = \{G \in \mathrm{M}(n, K_p)\colon\ e(G) = \mathrm{e}(n, K_p) \text{ and } G \text{ is non-}(p-1)\text{-partite}\}.$$

We will study possible size of the maximal K_p-free graphs with n vertices. This problem for $p = 3$ was solved in [11]. The same result was obtained in [3]. In these papers the minimum and maximum size of maximal K_3-free graphs was determined. Moreover, it was proved there that for every integer m such that $\mathrm{s}(n, K_3) \leq m \leq \mathrm{e}(n, K_3)$ there exists a maximal K_3-free graph with size m and with n vertices. In Section 2 we will deal with the K_3-free graphs, we will recall some theorems and we will give the stronger result: we completely characterize the set $\mathrm{E}(n, K_3)$. The case for $p = 4$ was studied in [2]. In Sections 3, 4, 5 we will deal with the maximal K_p-free graphs for $p \geq 4$. We will determine the minimum and maximum size of n-vertex non-$(p-1)$-partite maximal K_p-free graphs. In Section 4 we completely determine the set $\mathrm{E}(n, K_p)$. In Section 5 we will solve the interpolation problem.

2. Maximal K_3-free Graphs

Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_k\}$ and n_i be integers for $i = 1, \ldots, k$. By $G[n_1, \ldots, n_k]$ we denote the graph of order $n_1 + \cdots + n_k$ obtained from G in the following way: each vertex v_i we replaced by the set V_i of n_i independent vertices for $i = 1, \ldots, k$. We join each vertex of V_i with each vertex of V_j whenever vertices v_i and v_j are adjacent in the graph G.

Murty [10] characterized 2-connected graphs with diameter 2 with the minimum number of edges. Let P be the Petersen graph and G_7 be the graph in Figure 1.

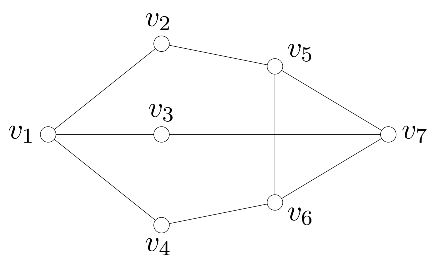

Figure 1. The graph G_7.

Theorem 1 [10]**.** *Let G be a 2-connected graph of order n such that* $\mathrm{diam}(G) = 2$ *with the minimum size. Then $e(G) = 2n-5$ and $G \in \{C_5[t, 1, n-t-3, 1, 1] : 1 \leq t \leq n-4\} \cup \{G_7[1, t_1, t_2, n-t_1-t_2-4, 1, 1, 1] : t_1, t_2 \geq 1, t_1+t_2 \leq n-5\} \cup \{P\}$.*

Lemma 2. *Let G be a non-bipartite maximal K_3-free graph. Then G is 2-connected and* $\operatorname{diam}(G) = 2$.

Proof. Suppose that there are two vertices $u, v \in V(G)$ such that $d_G(u, v) > 2$, where $d_G(u, v)$ denotes the distance between u and v. Thus, $G + uv$ does not contain K_3, so G is not maximal. This yields that $\operatorname{diam}(G) = 2$. Since G is not bipartite and $\operatorname{diam}(G) = 2$, we have that G is 2-connected. ■

From Theorem 1 and Lemma 2 it immediately follows

Theorem 3. *Let $n \geq 5$. Then*
(a) $\mathrm{s}(n, K_3) = 2n - 5$,
(b) $\mathrm{S}(n, K_3) = \{C_5[t, 1, n - t - 3, 1, 1] : 1 \leq t \leq n - 4\} \cup \{P\}$.

For $n \geq 5$ let us denote $C_5^*[n] = \{C_5[n_1, \ldots, n_5] : n_1 + \cdots + n_5 = n\}$ and $C_5^* = \{C_5^*[n] : n \geq 5\}$. From Theorem 3 it follows that non-bipartite maximal K_3-free graphs of minimum size belong to C_5^*. In [7] it was proved that maximal K_3-free graphs with minimum degree 2 having minimum size belong to C_5^*. In the next theorem we will show that also non-bipartite K_3-free graphs with a maximum size belong to C_5^*. First we will show how to distribute the vertices in any graph from $C_5^*[n]$ to obtain the maximum size. Let us define the subclasses of $C_5^*[n]$:
for n even
$$A(n, K_3) = \{C_5[\tfrac{n}{2} - 2, k, 1, 1, \tfrac{n}{2} - k] : 1 \leq k \leq \tfrac{n}{2} - 1\},$$
$$B(n, K_3) = \{C_5[\tfrac{n}{2} - 1, k, 1, 1, \tfrac{n}{2} - k - 1] : 1 \leq k \leq \tfrac{n}{2} - 2\},$$
for n odd
$$C(n, K_3) = \{C_5[\tfrac{n-1}{2} - 1, k, 1, 1, \tfrac{n-1}{2} - k] : 1 \leq k \leq \tfrac{n-1}{2} - 1\}.$$

Lemma 4. *Let $n \geq 5$ and $G \in C_5^*[n]$ with the maximum size. Then*
$$G \in \begin{cases} A(n, K_3) \cup B(n, K_3) & \text{for } n \text{ even}, \\ C(n, K_3) & \text{for } n \text{ odd}. \end{cases}$$

Proof. Let $G = C_5[n_1, n_2, n_3, n_4, n_5]$. Let V_i $(i = 1, \ldots, 5)$ be independent sets of G such that $|V_i| = n_i$ $(i = 1, \ldots, 5)$. First we will show that in G there are two consecutive independent sets with exactly one vertex each. Let us consider two cases.

Case 1. There are two consecutive independent sets with distinct number of vertices. Without loss of generality we assume that V_1 and V_2 have distinct number of vertices and $n_1 > n_2$. We show that $n_3 = 1$. If this is not true (i.e., $n_3 \geq 2$), then we delete one vertex from V_3 and add one vertex to V_5, so we obtain the graph $C_5[n_1, n_2, n_3 - 1, n_4, n_5 + 1]$ having more edges than G, a contradiction. Now, we show that also $n_4 = 1$ or $n_2 = 2$. If $n_4 \geq 2$, we delete one vertex from V_4 and add one vertex to V_1. Hence we obtain $C_5[n_1 + 1, n_2, n_3, n_4 - 1, n_5]$

that has more edges than G if $n_2 \neq 1$. Thus, if G has a maximum size and two consecutive independent sets with distinct number of vertices, then it also has two consecutive independent sets with exactly one vertex each.

Case 2. All independent sets have the same number of vertices. Thus, $n_1 = n_2 = n_3 = n_4 = n_5 = p$. Suppose that $p \geq 2$. If we delete one vertex from V_2 and add one vertex to V_1 and we delete one vertex from V_3 and add one vertex to V_5, then we obtain a graph $C_5[p+1, p-1, p-1, p, p+1]$ with more edges.

Hence we may assume that $n_3 = n_4 = 1$. Then $e(G) = n_1(n - n_1 - 2) + n - n_1 - 2 + 1$. When n is even, $e(G)$ achieves the maximum for $n_1 = \frac{n}{2} - 1$ or $n_1 = \frac{n}{2} - 2$. When n is odd, $e(G)$ achieves the maximum for $n_1 = \frac{n-1}{2} - 1$. Thus, $G \in A(n, K_3) \cup B(n, K_3)$ for n even or $G \in C(n, K_3)$ for n odd. ■

Theorem 5. *Let n, q, r be integers such that $n \geq 5,\ n = 2q + r,\ r = 0, 1$. Then*

(a) $\mathrm{e}(n, K_3) = \frac{n^2}{4} - \frac{n}{2} + \frac{r}{4} + 1$,

(b) $\mathrm{E}(n, K_3) = \begin{cases} A(n, K_3) \cup B(n, K_3) & \textit{for } n \textit{ even}, \\ C(n, K_3) & \textit{for } n \textit{ odd}. \end{cases}$

Proof. Since (b) implies (a), we prove only the part (b). Let G be a graph with n vertices such that G is a non-bipartite K_3-free of maximum size, i.e., $G \in \mathrm{E}(n, K_3)$. First we show that $G \in C_5^*[n]$. Next we use Lemma 4 to obtain (b). Let v be the vertex of maximum degree $d(v) = \Delta(G) = \Delta$. Since G is triangle-free, $N(v)$ is an independent set and since G is not a bipartite G contains an odd cycle of order at least 5. Hence $G - N[v]$ contains at least one edge. Suppose that $G - N[v]$ contains two vertex-disjoint edges xy and $x'y'$. Consider deleting all edges adjacent to x' and all edges adjacent to y' and next we join vertices x' and y' with all vertices of $N(v)$. Since $|N(v)| = \Delta(G)$, this new graph has more edges than G and it is a K_3-free graph, a contradiction. Hence $G - N[v]$ does not contain two vertex-disjoint edges, so $G - N[v] = S_{t+1} \cup \overline{K}_{n-\Delta-t-2}$.

First suppose that the graph $G - N[v]$ has exactly one edge xy. Let X and Y be the sets of neighbours in $N(v)$ of x and y, respectively. The set $X \cap Y = \emptyset$ because G is K_3-free and $X \cup Y = N(v)$ because G is maximal. Also, neither X nor Y can be empty. For any vertex $z \in (V(G) \setminus N[v]) \setminus \{x, y\}$ we have $N(z) = N(v)$. This implies that we can divide $V(G)$ into five independent sets $V_1 = \{x\},\ V_2 = \{y\},\ V_3 = Y,\ V_4 = (V(G) \setminus N(v)) \setminus \{x, y\},\ V_5 = X$ such that the sets $V_i \cup V_j,\ j = i + 1 \pmod 5$, induce a complete bipartite graph. Thus, $G \in C_5^*[n]$.

Now suppose that $t \geq 2$. Let us denote by $x, x_1, x_2, \ldots, x_t$ vertices of S_{t+1} such that x is a central vertex of the star. Since $N(v) = \Delta(G)$, each vertex x_i $(i = 1, \ldots, t)$ is nonadjacent to at least one vertex of $N(v)$. Suppose that there is j such that x_j is nonadjacent to more than one vertex in $N(v)$. We can delete the edge xx_j and join x_j with all vertices of $N(v)$. The new graph has more edges than G and is K_3-free, a contradiction. Thus, each vertex x_i $(i = 1, \ldots, t)$ is nonadjacent to exactly one vertex in $N(v)$. By Lemma 2 diam$(G) = 2$ and hence x has a neighbour w in $N(v)$. Since G is K_3-free, w is nonadjacent to all neighbours of x. Thus, all vertices x_i $(i = 1, \ldots, t)$ are nonadjacent to the vertex w. Therefore, we can divide $V(G)$ into the following independent sets: $V_1 = \{x\}$, $V_2 = \{x_1, \ldots, x_t\}$, $V_3 = N(v) \setminus \{w\}$, $V_4 = V(G) \setminus (N(v) \cup V(S_{t+1}))$, $V_5 = \{w\}$. Thus, $G \in C_5^*[n]$, so by Lemma 4 we obtain (b). ■

For convenience we repeat the following result given in [3, 11].

Theorem 6. *Let n, q, r be integers such that $n \geq 5$, $n = 2q + r$, $r = 0, 1$. Then for any integer m such that $2n - 5 \leq m \leq \frac{n^2}{4} - \frac{n}{2} + \frac{r}{4} + 1$ there is a maximal K_3-free graph of size m with n vertices.*

Proof. If $n = 5$ then $m = 2n - 5 = \frac{n^2}{4} - \frac{n}{2} + \frac{r}{4} + 1 = 5$ and C_5 is the only graph in $\mathrm{M}(5, K_3)$. For $n \geq 6$, let $G_t^x(n) = C_5[t, 1, x - t, \frac{n-x}{2} - 1, \frac{n-x}{2}]$ where $2 \leq x \leq n - 4$, $1 \leq t \leq x - 1$ and x, n are the same parity. It is easy to see that $G_t^x(n) \in \mathrm{M}(n, K_3)$ and $e(G_t^x(n)) = \frac{n^2-x^2}{4} + \frac{x-n}{2} + t$. Moreover, $e(G_1^{n-4}) = 2n - 5$ and $e(G_1^2(n)) = \frac{n^2}{4} - \frac{n}{2} + 1$ for $n = 2q$, $e(G_1^3(n)) = \frac{n^2}{4} - \frac{n}{2} + \frac{1}{4} + 1$ for $n = 2q + 1$.

Let $x = n - 4$ and $t = 1$. If we increase t by 1, then we obtain the graph with one extra edge. If we decrease x by 2, then we obtain the graph with $x - 2$ extra edges, i.e, $e(G_{t+1}^x(n)) = e(G_t^x(n)) + 1$ and $e(G_t^{x-2}(n)) = e(G_t^x(n)) + x - 2$.

Thus, if we fix x and increase t by 1 from $t = 1$ to $t = x - 1$, then we obtain the sequence of graphs whose sizes are all integers from the interval $[\frac{n^2-x^2}{4} + \frac{x-n}{2} + 1, \frac{n^2-x^2}{4} + \frac{x-n}{2} + t]$. Next, if we decrease the value of x by 2 from $x = n - 4$ to $x = 2$ for n even and to $n = 3$ for n odd, then we obtain all integers m from the interval $[2n - 5, \frac{n^2}{4} - \frac{n}{2} + \frac{r}{4} + 1]$. ■

3. Minimum Size of Non-$(p-1)$-partite Maximal K_p-free Graphs

The theorem of Erdős, Hajnal and Moon [5] states that if the graph G is maximal K_p-free, then $e(G) \geq (p - 2)n - \frac{1}{2}(p - 1)(p - 2)$ and the bound is realized by the complete $(p - 1)$-partite graph $K_{1,1,\ldots,1,n-p+2}$. The next complete $(p - 1)$-partite graph $K_{1,1,\ldots,2,n-p+1}$ has $(p - 1)n - \frac{1}{2}(p - 1)p - 1$ edges. We will show that the minimum size of non-$(p - 1)$-partite maximal K_p-free graphs with n vertices is $(p - 1)n - \frac{1}{2}(p - 1)p - 2$ if n is large enough.

We need the following results.

Theorem 7 [9]**.** *If $G \in \mathrm{M}(n, K_p)$ and G contains no vertex of degree $n - 1$, then $\delta \geq 2(p - 2)$*

Theorem 8 [1]**.** *Let $G \in \mathrm{M}(n, K_4)$ and $\delta(G) = 4$. If G contains no vertex of degree $n - 1$, then $e(G) \geq 4n - 15$.*

Theorem 9. *Let p, n be integers such that $p \geq 3$ and $n \geq 3(p+4)$. Then*

$$\mathrm{s}(n, K_p) = (p-1)n - \frac{1}{2}(p-1)p - 2.$$

Proof. Let $G = F + K_{p-3}$, $F \in \mathrm{S}(n-(p-3), K_3)$. Thus, the graph G is K_p-maximal non-$(p-1)$-partite and $e(G) = (p-1)n - \frac{1}{2}(p-1)p - 2$. Hence $\mathrm{s}(n, K_p) \leq (p-1)n - \frac{1}{2}(p-1)p - 2$.

Now we show that $\mathrm{s}(n, K_p) \geq (p-1)n - \frac{1}{2}(p-1)p - 2$. We prove by induction on p. By Theorem 3, the result holds for $p = 3$. Assume that the result holds for $p-1$, i.e. $\mathrm{s}(n, K_{p-1}) \geq (p-2)n - \frac{1}{2}(p-2)(p-1) - 2$. Let $G \in \mathrm{S}(n, K_p)$. Suppose that $\Delta(G) = n-1$. Let v be the vertex of degree $n-1$. Since G is maximal K_p-free, $G - v$ is maximal K_{p-1}-free. The assumption that G is not $(p-1)$-partite implies that $G - v$ is not $(p-2)$-partite. Thus, by the induction hypothesis

$$e(G-v) \geq (p-2)n - \tfrac{1}{2}(p-2)(p-1) - 2,$$

hence

$$e(G) = e(G-v) + n - 1 \geq (p-1)n - \tfrac{1}{2}(p-1)p - 2.$$

Thus, we may assume that $\Delta(G) \leq n-2$. Then by Theorem 7 we have $\delta(G) \geq 2(p-2)$. If $\delta(G) \geq 2(p-1)$, then $e(G) \geq (p-1)n$. Thus, to complete the proof we consider $\delta(G) = 2(p-2)$ and $\delta(G) = 2p-3$.

Let v be a vertex with minimum degree and let $H = V(G) \setminus N[v]$. Since G is maximal, for any vertex $x \in H$ the subgraph $G[N(x) \cap N(v)]$ contains K_{p-2}. Let

$$T = \{y \in N(v) : y \text{ is in a } (p-2)\text{−clique of } G[N(v)]\}.$$

Let $|T| = t$. Each vertex of H has at least $p-2$ neighbours in T and each vertex of T has at least $p-3$ neighbours in T. Thus,
$e(G[T \cup H]) \geq \frac{1}{2}t(p-3) + \sum_{x \in H} d_T(x)$.
Moreover,
$|E(G-v) \setminus E(G[T \cup H])| \geq \sum_{x \in N(v) \setminus T} d_{T \cup H}(x) + \frac{1}{2}\sum_{x \in N(v) \setminus T}(d_G(x) - 1$
$- d_{T \cup H}(x)) + \frac{1}{2}\sum_{x \in H}(d_G(x) - d_T(x))$
$= \frac{1}{2}\sum_{x \in N(v) \setminus T}(d_G(x) - 1 + d_{T \cup H}(x)) + \frac{1}{2}\sum_{x \in H}(d_G(x) - d_T(x))$
$\geq \frac{1}{2}\sum_{x \in N(v) \setminus T}(d_G(x) - 1) + \frac{1}{2}\sum_{x \in H}(d_G(x) - d_T(x))$.
Now we can calculate the lower bound for $e(G)$. Let $\delta(G) = \delta$.
$e(G) = e(G[T \cup H]) + |N(v)| + |E(G-v) \setminus E(G[T \cup H])|$
$\geq \frac{1}{2}t(p-3) + \sum_{x \in H} d_T(x) + \delta + \frac{1}{2}\sum_{x \in N(v) \setminus T}(d_G(x) - 1) + \frac{1}{2}\sum_{x \in H}(d_G(x) - d_T(x))$
$= \frac{1}{2}t(p-3) + \delta + \frac{1}{2}\sum_{x \in N(v) \setminus T}(d_G(x) - 1) + \frac{1}{2}\sum_{x \in H}(d_G(x) + d_T(x))$
$\geq \frac{1}{2}t(p-3) + \delta + \frac{1}{2}(\delta - t)(\delta - 1) + \frac{1}{2}|H|(\delta + p - 2)$
$= \frac{1}{2}t(p-2-\delta) + \frac{1}{2}\delta(\delta-1) + \frac{1}{2}(n-1-\delta)(\delta+p-2)$.

Since $\delta(G) = 2(p-2)$ or $\delta(G) = 2p-3$, this expression has the smallest value when t is as large as possible. Since $t \leq \delta$, we have $e(G) \geq -\frac{1}{2}\delta^2 - \delta + \frac{1}{2}\delta n + \frac{1}{2}(n-1)(p-2)$. When $\delta(G) = 2(p-2)$, we have $-\frac{1}{2}\delta^2 - \delta + \frac{1}{2}\delta n + \frac{1}{2}(n-1)(p-2) \geq$

$(p-1)n - \frac{1}{2}(p-1)p - 2$ for $p \geq 5$ and $n \geq 3(n+4)$. When $\delta(G) = 2p-3$, we have $-\frac{1}{2}\delta^2 - \delta + \frac{1}{2}\delta n + \frac{1}{2}(n-1)(p-2) \geq (p-1)n - \frac{1}{2}(p-1)p - 2$ for $p \geq 4$ and $n \geq 3(n+4)$. To complete the proof note that by Theorem 8 $\mathrm{s}(n, K_p) \geq (p-1)n - \frac{1}{2}(p-1)p - 2$ for $p = 4$ and $\delta(G) = 4$. ■

4. Maximum Size of Non-$(p-1)$-partite K_p-free Graphs

In this section we will give a maximum size of the non-$(p-1)$-partite K_p-free graphs for $p \geq 4$. We will also determine the set $E(n, K_p)$ for $p \geq 4$. We will prove this in the following way. First we will show that the non-$(p-1)$-partite K_p-free graph G of maximum size is the join of the non-bipartite K_3-free graph of maximum size with the $(p-3)$-partite graph, i.e., $G = G_1 + G_2$, where $G_1 \in C_5^*$ and G_2 is complete $(p-3)$-partite. Next we will show how to distribute the vertices of G between G_1 and G_2 to obtain a maximum size.

We need the following lemma.

Lemma 10. *Let G be a maximal K_p-free graph and $v \in V(G)$. Let xy be such an edge that, $x, y \in V(G) \setminus N[v]$. Then the vertices $N(v) \cap N(x) \cap N(y)$ induce the K_{p-2}-free graph and $|N(v) \setminus (N(x) \cap N(y))| \geq 2$.*

Proof. If the subgraph induced by $N(v) \cap N(x) \cap N(y)$ had a clique K_{p-2}, this clique together with x and y would form K_p. Since G is the maximal K_p-free graph, the subgraph $N(v) \cap N(x)$ contains a clique K' on $p-2$ vertices and also the subgraph $N(v) \cap N(y)$ contains a clique K'' on $p-2$ vertices. If $K' = K''$, then this clique together with x, y form K_p, a contradiction. Thus, at least one vertex of K' is not adjacent to y and at least one vertex of K'' is not adjacent to x. ■

Let us introduce the following notations. For $S \subseteq V(G)$, $e(S)$ denotes the number of edges incident with vertices of S, i.e., $e(S) = e(G[N[S]])$. For $S_1, S_2 \in V(G)$, by the symbol $E(S_1, S_2)$ we denote the set of all edges linking a vertex from the set S_1 with a vertex from the set S_2, i.e., $E(S_1, S_2) = \{uv \in E(G) : u \in S_1, v \in S_2\}$. Let $e(S_1, S_2) = |E(S_1, S_2)|$. Let $T_p^* = \{T_p(n) : n \geq p\}$.

Theorem 11. *Let $p \geq 3$ and $n \geq p+2$. If $G \in \mathrm{E}(n, K_p)$, then $G = G_1 + G_2$ where $G_1 \in C_5^*$ and G_2 is complete $(p-3)$-partite.*

Proof. Let v be the vertex of maximum degree and $\Delta(G) = \Delta$. We consider two cases.

Case 1. $G[N(v)]$ is not $(p-2)$-partite. We prove by induction on p. For $p = 3$ the proof follows from Theorem 5. Suppose that the subgraph induced by $V(G) \setminus N[v]$ contains an edge. Since $|N(v)| = \Delta$, if we delete all the edges in

$G[V(G) \setminus N[v]]$ and join each vertex of $V(G) \setminus N[v]$ to all vertices of $N(v)$, then we obtain a non-$(p-1)$-partite K_p-free graph with more edges, a contradiction. Thus, $V(G) \setminus N[v]$ is the independent set of vertices. Since $G[N(v)]$ is K_{p-1}-free and is not $(p-2)$-partite, we have by the induction hypothesis that $G[N(v)] = G_1 + G_2'$, where $G_1 \in C_5^*$ and G_2' is complete $(p-4)$-partite. This implies that G_2' together with $V(G) \setminus N(v)$ form the complete $(p-3)$-partite graph G_2. Therefore, $G = G_1 + G_2$ where $G_1 \in C_5^*$ and G_2 is complete $(p-3)$-partite.

Case 2. $G[N(v)]$ is $(p-2)$-partite. Let $H = V(G) \setminus N[v]$. Since the graph G is not $(p-1)$-partite, there is an edge in the subgraph induced by H. Let $x, y \in H$ and $xy \in E(G)$. Let S be the maximum K_{p-2}-free subgraph of $G[N(v)]$ (i.e. K_{p-2}-free with maximum number of vertices) and $|S| = s$. Since $G[N(v)]$ contains K_{p-2}, we have $\Delta - s \geq 1$. Let us consider two cases.

Subcase 2.1. $\Delta - s \geq 2$. Let $F = G_1 + G_2$, where $G_1 = C_5[1, 1, 1, \Delta - s - 1, n - \Delta - 2]$ and $G_2 \in T_{p-3}(s)$. Note that $e(F) = n\Delta - \Delta^2 - s^2 + \Delta s - \Delta + s + 1 + e(T_{p-3}(s))$ and F is non-$(p-1)$-partite K_p-free. Since $G \in E(n, K_p)$, it follows that $e(G) \geq n\Delta - \Delta^2 - s^2 + \Delta s - \Delta + s + 1 + e(T_{p-3}(s))$.

On the other hand we can calculate the size of G in the following way: $e(G) = d(v) + e(H \setminus \{x, y\}) + e(\{x, y\}, N(v)) + e(G[N(v) \setminus S]) + e(N(v) \setminus S, S) + 1 + e(G[S])$.

Note that $e(H \setminus \{x, y\}) \leq (|H| - 2)\Delta$. The subgraph induced by $N(v) \cap N(x) \cap N(y)$ is K_{p-2}-free, this yields that $|N(v) \cap N(x) \cap N(y)| \leq s$, since s is order of the maximum K_{p-2}-free subgraph of $G[N(v)]$. Thus, $e(\{x, y\}, N(v)) \leq \Delta + s$. The subgraph induced by $N(u) \cap N(v)$ for any $u \in N(v) \setminus S$ is K_{p-2}-free, since otherwise the subgraph induced by $N[v]$ would contain K_p. Thus, $e(G[N(v) \setminus S]) + e(N(v) \setminus S, S) \leq (\Delta - s)s$. Therefore, $e(G) \leq \Delta + (|H| - 2)\Delta + \Delta + s + (\Delta - s)s + 1 + e(G[S]) \leq n\Delta - \Delta^2 - s^2 + \Delta s - \Delta + s + 1 + e(T_{p-3}(s))$.
We conclude that we obtain the graph of maximum size if the equality holds. This implies the following:

(1) Each vertex of $H \setminus \{x, y\}$ has maximum degree.
(2) The set $H \setminus \{x, y\}$ is independent.
(3) The vertices $N(v) \cap N(x) \cap N(y)$ induce the maximum K_{p-2}-free subgraph of $G[N(v)]$.
(4) $N(v) \subseteq N(x) \cup N(y)$.
(5) Each vertex of $N(v) \setminus S$ is adjacent to all vertices of S.
(6) The vertices of S induce a graph from T_{p-3}^*.

From (5) and (6) it immediately follows

Claim 1. $G[N(v)]$ is the complete $(p-2)$-partite graph.

Since $G[N(v)]$ is the complete $(p-2)$-partite graph and $N(v) \cap N(x) \cap N(y)$ induces the maximum K_{p-2}-free subgraph of $G[N(v)]$ (by (3)), we have the following

Claim 2. The vertices of $N(v) \cap N(x) \cap N(y)$ induce the complete $(p-3)$-partite graph.

Let G_2 be the subgraph of G induced by $N(v) \cap N(x) \cap N(y)$, so G_2 is complete $(p-3)$-partite by Claim 2.

Claim 3. Each vertex of $V(G) \setminus V(G_2)$ is adjacent to all vertices of $V(G_2)$.

Proof. It is easy to see that each vertex of $(V(G) \setminus V(G_2)) \setminus (H \setminus \{x, y\})$ is adjacent to all vertices of $V(G_2)$. Now we show that it holds also for each vertex of $H \setminus \{x, y\}$. First note that each vertex of $z \in H \setminus \{x, y\}$ is nonadjacent to at most two vertices of $N(v)$, since $d_G(z) = \Delta$ and $H \setminus \{x, y\}$ is independent (by (1) and (2)). Suppose that there is a vertex $z \in H \setminus \{x, y\}$ that is nonadjacent to a vertex of $V(G_2)$. First assume that z is nonadjacent to exactly one vertex of $N(v)$ (i.e., a vertex of G_2). Thus, z is adjacent either to x or to y. Since G is maximal K_p-free, $N(v) \cap N(z)$ must contain a clique on $p-2$ vertices. Since $G[N(v)]$ is complete $(p-2)$-partite, both $N(z) \cap N(x) \cap N(v)$ and $N(z) \cap N(y) \cap N(v)$ contains a $(p-2)$-clique. This implies that this clique either with z, x or z, y form K_p, a contradiction. Now assume that z is nonadjacent to exactly two vertices of $N(v)$ (at least one of them is in $V(G_2)$). Thus, z is adjacent to both x and y. Thus, either $N(z) \cap N(x)$ or $N(z) \cap N(y)$ contains K_{p-2}, so G contains K_p, a contradiction. □

To finish the proof of this case it is enough to see that vertices of $G \setminus V(G_2)$ must induce the K_3-free graph that is not bipartite. Moreover, since G has a maximum size $G_1 = G \setminus V(G_2) \in C_5^*$. Hence $G = G_1 + G_2$, where $G_1 \in C_5^*$ and G_2 is $(p-3)$-partite.

Subcase 2.2. $\Delta - s = 1$. Let $F = G_1 + G_2$, where $G_1 = C_5[1, 1, 1, 1, n - \Delta - 2]$ and $G_2 \in T_{p-3}(\Delta - 2)$. Note that $e(F) = (n - \Delta)\Delta + 3(\Delta - 2) + e(T_{p-3}(\Delta - 2))$. Thus,

$$(*) \qquad e(G) \geq n\Delta - \Delta^2 + 2\Delta - 5 + e(T_{p-3}(\Delta - 2)).$$

Let $w = N(v) \setminus S$. Since S is K_{p-2}-free, every $(p-2)$-clique of $G[N(v)]$ contains w. From fact that $N(x) \cap N(v)$ and $N(y) \cap N(v)$ contain K_{p-2}, we have $wx \in E(G)$ and $wy \in E(G)$. Since $d_G(w) \leq \Delta$, we have $d_S(w) \leq s - 2$. Let $u \in S$ such that $wu \notin E(G)$. Let $S' = S \setminus \{u\}$. We can calculate the size of G in the following way $e(G) = d(v) + e(H \setminus \{x, y\}) + e(\{x, y\}, N(v)) + e(\{w, u\}, S') + 1 + e(G[S'])$.

Since $\Delta(G) = \Delta$, $e(H \setminus \{x, y\}) \leq (|H| - 2)\Delta$. By Lemma 10, $e(\{x, y\}, N(v)) \leq 2\Delta - 2$. Since w is nonadjacent to two vertices of S, $e(\{w, u\}, S') \leq 2\Delta - 5$. Thus, $e(G) \leq \Delta + (n - \Delta - 3)\Delta + 2\Delta - 2 + 2\Delta - 5 + 1 + e(T_{p-3}(\Delta - 2)) = n\Delta - \Delta^2 + 2\Delta - 6 + e(T_{p-3}(\Delta - 2))$. But this contradicts $(*)$. ■

In the next lemma we show how to distribute the edges in the graph $G = G_1 + G_2$ such that $G_1 \in C_5^*$ and G_2 is a complete $(p-3)$-partite graph to obtain the maximum size.

Lemma 12. *Let $p \geq 4$ and $n \geq p+2$, $n = (p-1)q + r$, $(r = 0, 1, \ldots, p-2)$. Let $G = G_1 + G_2$ be the n-vertex graph such that $G_1 \in C_5^*$ and G_2 is a complete $(p-3)$-partite graph. If the graph G has the maximum size, then the following conditions hold:*

$$(1) \begin{cases} for\ q = 1, 2, \quad v(G_1) = 5, \\ for\ q \geq 3, \qquad v(G_1) \in \begin{cases} \{2q-1, 2q\} & for\ r = 0, \\ \{2q-1, 2q, 2q+1\} & for\ r = 1, 2, \ldots, p-4, \\ \{2q, 2q+1\} & for\ r = p-3, \\ \{2q+1\} & for\ r = p-2, \end{cases} \end{cases}$$

(2) *G_1 is the graph of maximum size in C_5^* and $G_2 \in T_{p-3}^*$.*

Proof. Because the number of edges in $E(V(G_1), V(G_2))$ depends neither on the structure of G_1 nor on the structure of G_2, it is easy to see that the condition (2) is satisfied for a graph of maximum size. We show that the condition (1) also holds. We prove this in the following way: if G_1 has less vertices than in thesis, then we delete a vertex from G_2 and add it to a proper set of G_1 and we show that the resulting graph has more edges; if G_1 has more vertices than in thesis, then we delete a vertex from G_1 and add it to a proper set of G_2 and we show that the new graph has more edges.

Let $v(G_1) = t$ and $A(t, K_3), B(t, K_3), C(t, K_3)$ be the families of graphs defined in Section 2. From Lemma 4 it immediately follows

Claim 4. If t is odd, then $G_1 \in C(t, K_3)$ and in G_1 there is a vertex x such that x is nonadjacent to $\frac{t-1}{2} + 1$ vertices of G_1.

Claim 5. If t is even, then $G_1 \in A(t, K_3) \cup B(t, K_3)$ and in G_1 there is a vertex x such that x is nonadjacent to $\frac{t}{2} + 1$ vertices.

Claim 6. If t is odd, then $G_1 \in C(t, K_3)$ and we can add a new vertex x and join x with $\frac{t-1}{2}$ vertices of G_1 in such a way that the resulting graph is in C_5^*.

Claim 7. If t is even, then $G_1 \in A(t, K_3) \cup B(t, K_3)$ and we can add a new vertex x and join x with $\frac{t}{2}$ vertices of G_1 in such a way that the resulting graph is in C_5^*.

For $q = 1, 2$, we have $p+2 \leq n \leq 3(p-1)-1$. It is easy to see that the result holds for $n = p+2$. Assume that $n \geq p+3$ and $v(G_1) \geq 6$. Let $V_1, V_2, \ldots, V_5$ be the independent sets that replace the vertices of C_5 in G_1. Since $v(G_1) \geq 6$, at least one set of $V_1, V_2, \ldots, V_5$ has at least two vertices. Let x be the vertex in this set. Thus, $d(x) \leq n-4$. Since $n \leq 3p-4$ and $v(G_1) \geq 6$, we have $v(G_2) \leq 3n-10$. Hence, there is a partite set of G_2 that has less than three vertices. If we shift the vertex x to this set, then we obtain a graph with more edges, because now $d(x) \geq n-3$.

Now suppose that $q \geq 3$, $r = 0$ and $v(G_1) \leq 2q-2$. Thus, $v(G_2) \geq (p-3)q+2$ and there is a partite set of G_2 having more than q vertices. Let x be the vertex in this set, so $d(x) \leq n-q-1$. Claim 6 and Claim 7 imply that if we shift the vertex x to G_1, then we obtain a graph with more edges, because now $d(x) \geq n-q$.

Suppose that for $q \geq 3$, $r = 0$ and $v(G_1) \geq 2q+1$. By Claim 4 and Claim 5 there is a vertex x such that $d(x) \leq n-q-1$. Because $v(G_2) \leq (p-3)q-1$, the graph G_2 contains a partite set with at most $q-1$ vertices. If we shift the vertex x to this set, then we obtain a graph with more edges, now $d(x) \geq n-q$.

Assume that $q \geq 3$ and $r = 1, 2, \ldots, p-4$. If $v(G_1) \leq 2q-2$, then $v(G_2) \geq (p-3)q+r+2$. Thus, there is a partite set of G_2 having at least $q+1$ vertices. Let x be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 if we shift the vertex x to G_1, then we obtain a graph with more edges, because now the vertex x is adjacent to at least $n-q$ edges. If $v(G_1) \geq 2q+2$, then by Claim 4 and Claim 5 there is a vertex x such that $d(x) \leq n-q-2$. Since $v(G_2) \leq (p-3)q+r-2$, the graph G_2 contains a partite set with at most q vertices. If we shift the vertex x to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$.

Assume that $q \geq 3$ and $r = p-3$. If $v(G_1) \leq 2q-1$ then $v(G_2) \geq (p-3)(q+1)+1$. Thus, there is a partite set of G_2 having at least $q+1$ vertices. Let x be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 we can shift the vertex x to G_1 to obtain a graph with more edges, because then the vertex x is adjacent to at least $n-q$ edges. If $v(G_1) \geq 2q+2$, then by Claim 4 and Claim 5 there is a vertex x such that $d(x) \leq n-q-2$. Since $v(G_2) \leq (p-3)(q+1)-2$, the graph G_2 contains a partite set with at most q vertices. If we shift the vertex x to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$.

Assume that $q \geq 3$ and $r = p-2$. If $v(G_1) \leq 2q$, then $v(G_2) \geq (p-3)(q+1)+1$. Thus, there is a partite set of G_2 with at least $q+1$ vertices. Let x be a vertex in this set, so $d(x) \leq n-q-1$. By Claim 6 and Claim 7 if we shift the vertex x to G_1, then we obtain a graph with more edges, because now the vertex x is adjacent to at least $n-q$ edges. If $v(G_1) \geq 2q+2$, then by Claim 4 and Claim 5 there is a vertex x such that $d(x) \leq n-q-2$. Since $v(G_2) \leq (p-3)(q+1)-1$, the graph G_2 contains a partite set with at most q vertices. If we shift the vertex x to this set, then we obtain a graph with more edges, now $d(x) \geq n-q-1$. ■

Let us denote by G_i the graphs from Lemma 12 achieving the maximum size. Let G_i $(i = 1, \ldots, 4)$ be the graph of order $n = (p-1)q+r$, $0 \leq r \leq p-2$, $p \geq 4$ such that $G_i = G_{i1} + G_{i2}$, where

$G_{11} = C_5$, $\quad G_{12} = T_{p-3}(n-5)$,

$G_{21} \in \mathrm{E}(2q-1, K_3)$, $\quad G_{22} = T_{p-3}(q(p-3)+r+1)$,

$G_{31} \in \mathrm{E}(2q, K_3)$, $\quad G_{32} = T_{p-3}(q(p-3)+r)$,
$G_{41} \in \mathrm{E}(2q+1, K_3)$, $\quad G_{42} = T_{p-3}(q(p-3)+r-1)$.

Then, from Theorem 11 and Lemma 12 it immediately follows

Theorem 13. *Let p, n, q, r be integers such that $p \geq 4$, $n \geq p+2$, $n = (p-1)q + r$, $0 \leq r \leq p-2$. Then*

$$\mathrm{E}(n, K_p) = \begin{cases} \{G_1\} & for\ q = 1, 2, \\ \{G_2, G_3\} & for\ q \geq 3\ and\ r = 0, \\ \{G_2, G_3, G_4\} & for\ q \geq 3\ and\ r = 1, 2, \ldots, p-4, \\ \{G_3, G_4\} & for\ q \geq 3\ and\ r = p-3, \\ \{G_4\} & for\ q \geq 3\ and\ r = p-2. \end{cases}$$

Theorem 13 implies the following

Theorem 14. *Let p, n, q, r be integers such that $p \geq 3$, $n \geq p+2$, $n = (p-1)q + r$, $0 \leq r \leq p-2$. Then*

$$\mathrm{e}(n, K_p) = \tfrac{p-2}{2(p-1)} n^2 - \tfrac{1}{p-1} n + \tfrac{r(r+2)}{2(p-1)} - \tfrac{r}{2} + 1.$$

5. Size of Maximal K_p-free Graphs

Note that $e(K_{1,1,\ldots,1,n-p+2}) = sat(n, K_p)$ (the minimum size of the maximal K_p-free graph with n vertices) and $e(T_{p-1}(n)) = t_{p-1}(n)$. Since $e(K_{1,1,\ldots,2,n-p+1}) > (p-1)n - \frac{1}{2}(p-1)p - 2$, Theorem 9 implies that for large n there is no maximal K_p-free graph with n vertices and size m such that $sat(n, K_p) < m < \mathrm{s}(n, K_p)$. From Theorem 14 we have that for any pair n, m such that $\mathrm{e}(n, K_p) < m \leq t_{p-1}(n)$ each n-vertex maximal K_p-free graph with n edges is complete $(p-1)$-partite.

Theorem 15. *Let p, n be integers such that $p \geq 3$, $n \geq 3p+4$. Then for any integer m such that $\mathrm{s}(n, K_p) \leq m \leq \mathrm{e}(n, K_p)$ there is a maximal K_p-free graph with n vertices and size m.*

Proof. Let us consider the family of n-vertex graphs $\alpha(n) = \{H + Q : H \in C_5^*,\ Q \in T_{p-3}^*,\ v(H) + v(Q) = n\}$. Observe that every graph from $\alpha(n)$ is non-$(p-1)$-partite maximal K_p-free. Let $n = q(p-1) + r$, $0 \leq r \leq p-2$. If $v(Q) = p-3$ and $H \in S((q-1)(p-1) + r - 2, K_3)$, then $e(H+Q) = \mathrm{s}(n, K_p)$. If $v(Q) = (p-3)q + r$ and $H \in E(2q, K_p)$ or $v(Q) = (p-3)q + r - 1$ and $H \in E(2q+1, K_p)$, then $e(H+Q) = \mathrm{e}(n, K_p)$. Let $\alpha_b(n) \subseteq \alpha(n)$ such that $\alpha_b(n) = \{H + Q : H \in C_5^*,\ Q \in T_{p-3}^*,\ v(Q) = b,\ v(H) = n-b\}$. Note that for any graph from $\alpha_b(n)$ the number of edges adjacent to vertices of Q is constant. Let e_b be the number of edges adjacent to vertices of Q in the graph from $\alpha_b(n)$.

From Theorem 6 it follows that for any integer e such that

$$e \in [e_b + 2(n-b) - 5, e_b + \tfrac{1}{4}(n-b)^2 - \tfrac{1}{2}(n-b) + \tfrac{1}{4}r^2 + 1],$$

where $r \equiv n - b \pmod 2$
there is a graph in $\alpha_b(n)$ with size e. To complete the proof we show that for b such that $p - 3 \leq b \leq q(p-3) + r$ the inequality

$$e_b + \frac{1}{4}(n-b)^2 - \frac{1}{2}(n-b) + \frac{1}{4}r^2 + 2 \geq e_{b+1} + 2(n-b-1) - 5$$

holds. Or, equivalently,

$$e_{b+1} - e_b \leq \frac{1}{4}(n-b)^2 - \frac{5}{2}(n-b) + \frac{1}{4}r^2 + 9.$$

To prove this observe the following: if in $H + Q \in \alpha_b(n)$ we shift a vertex v from H to Q (to the independent set V_1 with the smallest number of vertices), then we must delete all edges joining v with V_1 and add all edges joining v with H to obtain a graph from $\alpha_{b+1}(n)$. Thus, $e_{b+1} - e_b = n - b - 1 - |V_1|$. To finish the proof we conclude that $n - b - 1 - |V_1| \leq \frac{1}{4}(n-b)^2 - \frac{5}{2}(n-b) + \frac{1}{4}r^2 + 9$. Indeed, when $b \geq 3(p-3)$, we have $|V_1| \geq 3$, so $n - b - 1 - |V_1| \leq n - b - 4 \leq \frac{1}{4}(n-b)^2 - \frac{5}{2}(n-b) + \frac{1}{4}r^2 + 9$. When $p - 3 \leq b \leq 3(p-3) - 1$, we have $n - b \geq 14$. Thus, $n - b - 1 - |V_1| \leq n - b - 2 \leq \frac{1}{4}(n-b)^2 - \frac{5}{2}(n-b) + \frac{1}{4}r^2 + 9$, for $n - b \geq 14$. ∎

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