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## COLORING SOME FINITE SETS IN $\mathbb{R}^n$

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### Abstract

This note relates to bounds on the chromatic number  $\chi(\mathbb{R}^n)$  of the Euclidean space, which is the minimum number of colors needed to color all the points in  $\mathbb{R}^n$  so that any two points at the distance 1 receive different colors. In [6] a sequence of graphs  $G_n$  in  $\mathbb{R}^n$  was introduced showing that  $\chi(\mathbb{R}^n) \geq \chi(G_n) \geq (1 + o(1))\frac{n^2}{6}$ . For many years, this bound has been remaining the best known bound for the chromatic numbers of some low-dimensional spaces. Here we prove that  $\chi(G_n) \sim \frac{n^2}{6}$  and find an exact formula for the chromatic number in the case of  $n = 2^k$  and  $n = 2^k - 1$ .

**Keywords:** chromatic number, independence number, distance graph.

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## 1. INTRODUCTION

In this note, we study the classical chromatic number  $\chi(\mathbb{R}^n)$  of the Euclidean space. The quantity  $\chi(\mathbb{R}^n)$  is the minimum number of colors needed to color all the points in  $\mathbb{R}^n$  so that any two points at a given distance  $a$  receive different colors. By a well-known compactness result of Erdős and de Bruijn (see [1]), the value of  $\chi(\mathbb{R}^n)$  is equal to the chromatic number of a *finite* distance graph  $G = (V, E)$ , where  $V \subset \mathbb{R}^n$  and  $E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = a\}$ .

Now we know that

$$(1.239 \dots + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n,$$

where the lower bound is due to the third author of this paper (see [8]) and the upper bound is due to Larman and Rogers (see [6]). Also, in [3] one can find an up-to-date table of estimates obtained for the dimensions  $n \leq 12$ .

It is worth noting that the linear bound  $\chi(\mathbb{R}^n) \geq n + 2$  is quite simple, and the first superlinear bound was discovered by Larman, Rogers, Erdős, and Sós in [6]. They considered a family of graphs  $G_n = (V_n, E_n)$  with

$$V_n = \{\mathbf{x} = (x_1, \dots, x_n) : x_i \in \{0, 1\}, x_1 + \dots + x_n = 3\},$$

$$E_n = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = 2\}.$$

In other words, the vertices of  $G_n$  are all the 3-subsets of the set  $[n] = \{1, \dots, n\}$  and two vertices  $A, B$  are connected with an edge iff  $|A \cap B| = 1$ . Larman and Rogers [6] used an earlier result by Zs. Nagy who proved the following theorem.

**Theorem 1** [6]. *Let  $s$  and  $t \leq 3$  be nonnegative integers and let  $n = 4s + t$ . Then*

$$\alpha(G_n) = \begin{cases} n, & \text{if } t = 0, \\ n - 1, & \text{if } t = 1, \\ n - 2, & \text{if } t = 2 \text{ or } t = 3. \end{cases}$$

The standard inequality  $\chi(G_n) \geq \frac{|V_n|}{\alpha(G_n)}$  combined with the above theorem gives an obvious corollary.

**Corollary 2** [6]. *Let  $s$  and  $t \leq 3$  be nonnegative integers and let  $n = 4s + t$ . Then*

$$\chi(G_n) \geq \begin{cases} \frac{(n-1)(n-2)}{6}, & \text{if } t = 0, \\ \frac{n(n-2)}{6}, & \text{if } t = 1, \\ \frac{n(n-1)}{6}, & \text{if } t = 2 \text{ or } t = 3. \end{cases}$$

The bounds from the corollary are applied to estimate from below the chromatic numbers  $\chi(\mathbb{R}^{n-1})$ , since the vertices of  $G_n$  lie in the hyperplane  $x_1 + \dots + x_n = 3$ . Now all these bounds are surpassed due to the consideration of some other distance graphs (see [3]). However, it could happen that actually  $\chi(G_n)$  is much bigger than the ratio  $\frac{|V_n|}{\alpha(G_n)}$ . It turns out that this is not the case, and the main result of this note is as follows.

**Theorem 3.** *If  $n = 2^k$  for some integer  $k \geq 2$ , then*

$$\chi(G_n) = \frac{(n-1)(n-2)}{6}.$$

*Additionally, if  $n = 2^k - 1$  for some integer  $k \geq 2$ , then*

$$\chi(G_n) = \frac{n(n-1)}{6}.$$

*Finally, there is a constant  $c$  such that for every  $n$ ,*

$$\chi(G_n) \leq \frac{(n-1)(n-2)}{6} + cn.$$

Our proof yields that  $c \leq 5.5$ . With some more work we could prove that  $c \leq 4.5$ . On the other hand, since  $n(n-1)/6 - (n-1)(n-2)/6 = (n-1)/3$ , we have  $c \geq 1/3$ .

In the next section, we prove Theorem 3.

## 2. PROOF OF THEOREM 3

Easily,  $\chi(G_3) = 1$ ,  $\chi(G_4) = 1$ ,  $\chi(G_5) = 3$ .

Let  $f(n) := \frac{(n-1)(n-2)}{6}$ . We show by induction on  $k$  that  $\chi(G_{2^k}) = f(2^k)$ . For  $k = 2$  it is trivial. Assume that for some  $k$  we established the induction hypothesis. Partition the set  $[n] = [2^{k+1}]$  into the equal parts  $A_1 = [\frac{n}{2}]$ ,  $A_2 = [n] \setminus [\frac{n}{2}]$  of size  $2^k$ . Denote by  $U_1$  and  $U_2$  the sets of vertices of  $G = G_{2^{k+1}}$  lying in the sets

$A_1$  and  $A_2$  respectively. By the induction assumption, each of the induced subgraphs  $G[U_1]$  and  $G[U_2]$  can be properly colored with at most  $f(2^k)$  colors. Cover all pairs of the  $2^k$  elements of  $A_1$  with disjoint perfect matchings  $N_1, \dots, N_{2^{k-1}}$  and all pairs of the  $2^k$  elements of  $A_2$  with matchings  $M_1, \dots, M_{2^{k-1}}$ . We form a color class  $C(i, j)$  for  $1 \leq i \leq 2^k - 1, 1 \leq j \leq 2^{k-1}$  as follows. Consider the matchings  $N_i, M_i$  and assume that the edges are  $\{u_1, u_2\}, \{u_3, u_4\}, \dots$  in  $N_i$  and  $\{v_1, v_2\}, \{v_3, v_4\}, \dots$  in  $M_i$ . For  $j = 1, \dots, 2^{k-1}$  let  $D(i, j)$  denote the following set of 4-tuples (indices are considered modulo  $2^k$ ):

$$\{u_1, u_2, v_{2j-1}, v_{2j}\}, \{u_3, u_4, v_{2j+1}, v_{2j+2}\}, \dots, \{u_{2^k-1}, u_{2^k}, v_{2j-3}, v_{2j-2}\}.$$

For  $i = 1, \dots, 2^k - 1$  and  $j = 1, \dots, 2^{k-1}$ , the color class  $C(i, j)$  is formed by the collection of triples contained in the members of  $D(i, j)$ . The intersection sizes are all 0 or 2, so the triples in  $C(i, j)$  form an independent set in  $G$ . Moreover, each triple is contained in a member of some  $D(i, j)$ . The total number of used colors is

$$2^{k-1}(2^k - 1) + f(2^k) = 2^{2k-1} - 2^{k-1} + \frac{(2^k-1)(2^k-2)}{6} = f(2^{k+1}).$$

This proves the first statement of the theorem. Since  $\chi(G_n) \leq \chi(G_{n+1})$ , this together with Corollary 2 also implies the statement of the theorem for  $n = 2^k - 1$ .

It remains to show that there exists a constant  $c$  such that  $\chi(G_n) \leq \frac{n^2}{6} + cn$  for every  $n$ . Consider our coloring in steps.

*Step 1:* Let  $n = 4s_1 + t_1$  where  $t_1 \leq 3$ . First, color all triples containing the elements  $4s_1 + 1, \dots, 4s_1 + t_1$  with at most  $t_1(n - 1) < 3n$  colors. Now consider the set  $[4s_1]$  and all the triples in this set. Partition  $[4s_1]$  into  $A_1 = [2s_1]$  and  $A_2 = [4s_1] - [2s_1]$  and color the triples intersecting both  $A_1$  and  $A_2$  with  $s_1(2s_1 - 1) < \frac{n}{4}(\frac{n}{2} - 1)$  colors as above.

*Step 2:* Since the triples contained in  $A_1$  are disjoint from the triples contained in  $A_2$ , we will use for coloring the triples contained in  $A_2$  the same colors and the same procedure as for the triples contained in  $A_1$ . Consider  $A_1$ . Let  $n_1 = |A_1| = 2s_1 = 4s_2 + t_2$  where  $t_2 \leq 3$ . Since  $2s_1$  is even,  $t_2 \leq 2$ . By construction,  $n_1 \leq \frac{n}{2}$ . Similarly to Step 1, color all triples containing the elements  $4s_2 + 1, \dots, 4s_2 + t_2$  with at most  $t_2(n_1 - 1) < 2n_1$  colors. Partition  $[4s_2]$  into  $A_{1,1} = [2s_2]$  and  $A_{1,2} = [4s_2] - [2s_2]$  and color the triples intersecting both  $A_{1,1}$  and  $A_{1,2}$  with at most  $\frac{n}{8}(\frac{n}{4} - 1)$  new colors.

*Step i (for  $i \geq 3$ ):* If  $2s_{i-1} \leq 2$ , then Stop. Otherwise, repeat Step 2 with  $[2s_{i-1}]$  in place of  $[2s_1]$ .

Altogether, we use at most  $(3n + \frac{n}{4}(\frac{n}{2} - 1)) + (\frac{2n}{2} + \frac{n}{8}(\frac{n}{4} - 1)) + (\frac{2n}{4} + \frac{n}{16}(\frac{n}{8} - 1)) + \dots < 5n + \frac{n^2}{8} \cdot \frac{4}{3} = \frac{n^2}{6} + 5n = \frac{(n-1)(n-2)}{6} + 5.5n - 1/3$  colors. The theorem is proved.  $\blacksquare$

3. DISCUSSION

As we have already said, the constant 5 in the bound  $\chi(G_n) \leq \frac{n^2}{6} + 5n$  is not the best possible and can be improved. However, to find the exact value of the chromatic number is still interesting. For example, we know that  $\chi(\mathbb{R}^{12}) \geq 27$  (see [3]). At the same time,  $\chi(G_{13}) \geq \left\lceil \frac{\binom{13}{3}}{12} \right\rceil = 24$  (due to Corollary 2), and the proof of Theorem 3 applied for  $n = 13$  yields a bound  $\chi(G_{13}) \leq 31$ .

It would be quite interesting to study more general graphs. Let  $G(n, r, s)$  be the graph whose set of vertices consists of all the  $r$ -subsets of the set  $[n]$  and whose set of edges is formed by all possible pairs of vertices  $A, B$  with  $|A \cap B| = s$ . Larman proved in [5] that

$$\chi(\mathbb{R}^n) \geq \chi(G(n, 5, 2)) \geq \frac{\binom{n}{5}}{\alpha(G(n, 5, 2))} \geq (1 + o(1)) \frac{\binom{n}{5}}{1485n^2} \sim \frac{n^3}{178200}.$$

Thus, the main result of Larman was in finding the bound  $\alpha(G(n, 5, 2)) \leq (1 + o(1))1485n^2$ . However, the so-called linear algebra method ([2], see also [8]) can be directly applied here to show that  $\alpha(G(n, 5, 2)) \leq (1 + o(1))\binom{n}{2} \sim \frac{n^2}{2}$ . This substantially improves Larman's estimate and gives  $\chi(G(n, 5, 2)) \geq (1 + o(1))\frac{n^3}{60}$ . We do not know any further improvements on this result. On the other hand, observe that for any 3-set  $A$ , the collection of 5-sets containing  $A$  forms an independent set in  $G(n, 5, 2)$ , yielding  $\chi(G(n, 5, 2)) \leq \binom{n}{3} \sim \frac{n^3}{6}$ . It is plausible that  $\chi(G(n, 5, 2)) \sim cn^3$  with a constant  $c \in [1/60, 1/6]$ , but this constant is not yet found and even no better bounds for  $c$  have been published.

Furthermore, the graphs  $G(n, 5, 3)$  have been studied, since the best known lower bound  $\chi(\mathbb{R}^9) \geq 21$  is due to the fact that  $\chi(G(10, 5, 3)) = 21$  (see [4]). No related results concerning the case of  $n \rightarrow \infty$  have apparently been published.

Although for combinatorial geometry small values of  $n$  are of greater interest, we see that the consideration of graphs  $G(n, r, s)$  with small  $r, s$  and *growing*  $n$  is of its intrinsic interest, too. So assume that  $r, s$  are fixed and  $n \rightarrow \infty$ . We have

$$\chi(G(n, r, s)) \leq \min\{O(n^{r-s}), O(n^{s+1})\}.$$

The first bound follows from Brooks' theorem, since the maximum degree of  $G(n, r, s)$  is

$$\binom{r}{s} \binom{n-r}{r-s} = (1 + o(1)) \frac{r!}{s!(r-s)!(r-s)!} n^{r-s}.$$

The second bound is a simple generalization of the above-mentioned bound  $\chi(G(n, 5, 2)) \leq (1 + o(1))n^3/6$ .

Note that the second bound can be somewhat improved. Assume  $s < r/2$ , so  $q := \lceil (r-1)/s \rceil$  is at least 2. Assuming that  $q$  divides  $n$ , partition  $[n]$  into  $q$

equal classes,  $A_1, \dots, A_q$ . Let  $\mathcal{C}$  be the family of  $(s+1)$ -sets that are subsets of some  $A_i$ . For each  $B \in \mathcal{C}$ , the  $r$ -sets containing  $B$  form an independent set in  $G(n, r, s)$ , and by the pigeonhole principle every  $r$ -set contains such  $B$ , hence

$$\chi(G(n, r, s)) \leq |\mathcal{C}| = q \binom{n/q}{s+1} = (1 + o(1)) \frac{n^{s+1}}{q^s (s+1)!}.$$

In particular,  $\chi(G(n, 5, 2)) \leq (1 + o(1)) \frac{n^3}{24}$ , which improves the previous bound  $\frac{n^3}{6}$ .

It is worthwhile to look at the construction in Section 2 from a different point of view. For  $n = 2^k$  we constructed a 4-uniform hypergraph  $\mathcal{H}$  with the property that every 3-subset of vertices is covered exactly once. Note that  $e(\mathcal{H}) = \binom{n}{3}/4$ . Then we decomposed  $E(\mathcal{H})$  into  $\binom{n}{3}$  perfect matchings. Each matching gives a color class of our coloring. Note that instead of providing the explicit decomposition, we could have used a classical theorem of Pippenger and Spencer [7], which claims the existence of  $(1 + o(1)) \binom{n}{3}$  covering matchings.

This motivates the following possible approach to the case  $r = 2s + 1$ . The discussion here is not a proof, it is just a sketch of a possible way to generalize our argument. Assume that we managed to construct an  $(r+s)$ -uniform hypergraph  $\mathcal{H}$  that covers every  $r$ -set exactly once. Then  $e(\mathcal{H}) = \binom{n}{r} / \binom{n}{r+s}$ . Assume that  $\mathcal{H}$  can be decomposed into  $t$  hypergraphs,  $\mathcal{N}_1, \dots, \mathcal{N}_t$ , such that for every  $i$  and every  $A, B \in \mathcal{N}_i$  we have  $|A \cap B| \leq s - 1$ . Then the  $r$ -sets covered by sets in  $\mathcal{N}_i$  form an independent set, yielding  $\chi(G(n, r, s)) \leq t$ . Probably a generalization of the theorem of Pippenger and Spencer [7] would give  $t \leq (1 + o(1)) \binom{n}{r} / \binom{n}{s} = (1 + o(1)) (s!/r!) n^{r-s}$ . This bound, if true, would be asymptotically best possible, since the already mentioned linear algebra method (see [2, 8]) ensures that  $\alpha(G(n, 2s+1, s)) \leq (1 + o(1)) \binom{n}{s}$  and so  $\chi(G(n, 2s+1, s)) \geq (1 + o(1)) \binom{n}{r} / \binom{n}{s}$ , provided  $s+1$  is a prime power. In particular, we would get  $\chi(G(n, 5, 2)) \sim \frac{n^2}{60}$ .

The case of simultaneously growing  $n, r, s$  has also been studied. Namely,  $r \sim r'n$  and  $s \sim s'n$  with any  $r' \in (0, 1)$  and  $s' \in (0, r')$  have been considered. This is due to the fact that the first exponential estimate to the quantity  $\chi(\mathbb{R}^n)$ ,  $\chi(\mathbb{R}^n) \geq (1.207 \dots + o(1))^n$ , was obtained by Frankl and Wilson in [2] with the help of some graphs  $G(n, r, s)$  having  $r \sim r'n$  and  $s \sim \frac{r'}{2}n$ . Lower bounds are usually based on the linear algebra (see [8]) and upper bounds can be found in [9].

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