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TREES WITH EQUAL 2-DOMINATION AND 2-INDEPENDENCE NUMBERS

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is a 2-dominating set if every vertex of $V - S$ is dominated at least 2 times, and S is a 2-independent set of G if every vertex of S has at most one neighbor in S. The minimum cardinality of a 2-dominating set a of G is the 2-domination number $\gamma_2(G)$ and the maximum cardinality of a 2-independent set of G is the 2-independence number $\beta_2(G)$. Fink and Jacobson proved that $\gamma_2(G) \leq \beta_2(G)$ for every graph G. In this paper we provide a constructive characterization of trees with equal 2-domination and 2-independence numbers.

Keywords: 2-domination number, 2-independence number, trees.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* $N(v)$ of a vertex v consists of the vertices adjacent to v, the *closed neighborhood* of v is defined by $N[v] = N(v) \cup \{v\}$ and $d_G(v) = |N(v)|$ is the *degree* of v. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. If u is a support vertex, then L_u will denote the set of leaves attached at u. We denote by $K_{1,t}$ a *star* of order $t + 1$. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with, respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A

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graph is G called a *corona* if it is constructed from a graph of H by adding for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv'.

In [4], Fink and Jacobson generalized the concepts of independent and dominating sets. Let k be a positive integer, a subset S of $V(G)$ is *k-independent* if the maximum degree of the subgraph induced by the vertices of S is less or equal to $k-1$. The subset S is *k-dominating* if every vertex of $V(G)-S$ has at least k neighbors in S. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of a k-dominating set and the *k-independence number* $\beta_k(G)$ is the maximum cardinality of a k-independent set. A minimum k-dominating set and a maximum k-independent set of a graph G is called a $\gamma_k(G)$-set and $\beta_k(G)$-set, respectively. Thus for $k=1$, the 1-independent and 1-dominating sets are the classical independent and dominating sets. A survey on k-domination and k-independence in graphs has been given by Chellali, Favaron, Hansberg and Volkmann and can be found in [2]. Also for more details on domination and its variations see the books of Haynes, Hedetniemi, and Slater [5, 6].

It is well known that every graph G satisfies $\gamma_1(G) \le \beta_1(G)$. In [4], Fink and Jacobson proved that $\gamma_2(G) \le \beta_2(G)$ and conjectured that for every graph G and positive integer k, $\gamma_k(G) \le \beta_k(G)$. The conjecture has been proved by Favaron [3] by showing that every graph G admits a set that is both a k-independent and a k-dominating. It follows from this stronger result that if G is a graph such that $\beta_k(G) = \gamma_k(G)$, then G has a set that is both $\gamma_k(G)$-set and $\beta_k(G)$-set. This useful property will be used in the proof of the main result. Note that trees T with $\gamma_1(T) = \beta_1(T)$ have been characterized in [1] by Borowiecki who proved that such trees must be either K_1 or coronas.

In this paper, we give a characterization of all trees T with equal 2-domination and 2-independence numbers. We will call such trees (γ_2, β_2)-trees. Note that the difference $\beta_2(G) - \gamma_2(G)$ can be arbitrarily large even for trees. To see this consider a tree T_j obtained from a path of order $2j+1$ where the vertices are labelled from 1 to $2j+1$ by attaching a path P_2 to each of the odd numbered vertices. Then $\beta_2(T_j) = 3j+2$ and $\gamma_2(T_j) = 2j+2$.

2. (γ_2, β_2)-trees

2.1. Observations

We give some useful observations.

Observation 1. *Every 2-dominating set of a graph G contains every leaf.*

Observation 2. *Let T be a non-trivial tree and $w \in V(T)$. Then $\gamma_2(T) \le \gamma_2(T-w)+1$.*

Proof. If D is a $\gamma_2(T-w)$-set, then $D \cup \{w\}$ is a 2-dominating set of T and hence $\gamma_2(T) \le |D| + 1$. ■

Observation 3. *Let T be a non-trivial tree and v a vertex of T. Then $\beta_2(T-v) \le \beta_2(T) \le \beta_2(T-v) + 1$.*

Proof. $\beta_2(T-v) \le \beta_2(T)$ follows from the fact that any 2-independent set of $T-v$ is also a 2-independent set of T. Now if D is $\beta_2(T)$-set, then $D - \{v\}$ is a 2-independent set of $T-v$ and hence $\beta_2(T-v) \ge |D| - 1$. ■

Observation 4. *Let T be a tree obtained from a nontrivial tree T' and a star $K_{1,p}$ of center vertex v by adding an edge vw at any vertex w of T'. Then,*
(1) *$\gamma_2(T') \le \gamma_2(T) - p$, with equality if either $p \ge 2$ or w is a leaf of T'.*
(2) *If $p \ge 2$, then $\beta_2(T) = \beta_2(T') + p$.*

Proof. **(1)** Let D be a $\gamma_2(T)$-set. Then by Observation 1, $L_v \subset D$ and, without loss of generality, $v \notin D$ (else substitute v by w in D). Then $D \cap V(T')$ 2-dominates T' and so $\gamma_2(T') \le |D \cap V(T')| = \gamma_2(T) - p$. Now if $p \ge 2$, then every $\gamma_2(T')$-set can be extended to a 2-dominating set of T by adding the p leaves of the added star, and hence $\gamma_2(T) \le \gamma_2(T') + p$. Assume now that $p = 1$ and let v' be the unique leaf adjacent to v. If w is a leaf in T', then w belongs to every $\gamma_2(T')$-set D' and $D' \cup \{v'\}$ is a 2-dominating set of T', implying that $\gamma_2(T) \le \gamma_2(T') + 1$. In both cases the equality is obtained.

(2) Let S' be any $\beta_2(T')$-set. Then clearly $S' \cup L_v$ is a 2-independent set of T, and so $\beta_2(T) \ge \beta_2(T') + |L_v|$. Now among all $\beta_2(T)$-sets, let S be one containing the maximum number of leaves. If there exists a leaf $v' \in L_v$ such that $v' \notin S$, then $v \in S$ (else $S \cup \{v'\}$ is a 2-independent set larger than S) but then $\{v'\} \cup S - \{v\}$ is a 2-independent set of T containing more leaves than S, a contradiction. Hence $L_v \subset S$ and so $S - L_v$ is a 2-independent set of T'. It follows that $\beta_2(T') \ge \beta_2(T) - |L_v|$ and the equality holds. ■

Observation 5. *Let T be a tree obtained from a nontrivial tree T' and a double star $S_{1,p}$ with support vertices u and v, where $|L_v| = p$ by adding an edge vw at a vertex w of T'. Then,*
(1) *$\beta_2(T) = \beta_2(T') + (p+2)$.*
(2) *$\gamma_2(T) \le \gamma_2(T') + (p+2)$, with equality if $\beta_2(T) = \gamma_2(T)$.*

Proof. **(1)** Let u' be the unique leaf neighbor of u and let S a $\beta_2(T)$-set containing the maximum number of leaves. Then as seen in the proof of Observation 4, $L_v \cup \{u'\} \subset S$. Also S contains either u or v for otherwise $S \cup \{u\}$ is a 2-independent set of T larger than S. Without loss of generality, $u \in S$ and so $S - (L_v \cup \{u, u'\})$ is a 2-independent set of T'. Hence $\beta_2(T') \ge \beta_2(T) - (|L_v| + 2)$.

The equality is obtained from the fact that every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding $L_v \cup \{u, u'\}$.

(2) Clearly if D' is a $\gamma_2(T')$-set, then $D' \cup L_v \cup \{u', v\}$ is a 2-dominating set of T and so $\gamma_2(T) \leq \gamma_2(T') + (p+2)$. Now assume that $\beta_2(T) = \gamma_2(T)$ and suppose that $\gamma_2(T) < \gamma_2(T') + (p+2)$. Then by item (1) we have

$$\beta_2(T') + (p+2) = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + (p+2),$$

implying that $\beta_2(T') < \gamma_2(T')$, a contradiction. Therefore if $\beta_2(T) = \gamma_2(T)$, then $\gamma_2(T) = \gamma_2(T') + (p+2)$. ■

Observation 6. *Let T be a tree obtained from a nontrivial tree T' and a path $P_3 = xyz$ by adding an edge xw at a vertex w of T'. Then*
(1) $\beta_2(T) = \beta_2(T') + 2$.
(2) $\gamma_2(T) \leq \gamma_2(T') + 2$, *with equality if* $\beta_2(T) = \gamma_2(T)$.

Proof. **(1)** If D' is a $\beta_2(T')$-set, then $D' \cup \{y, z\}$ is a 2-independent set of T and so $\beta_2(T) \geq \beta_2(T') + 2$. Now let D be a $\beta_2(T)$-set. Clearly $1 \leq |D \cap \{x, y, z\}| \leq 2$. If $|D \cap \{x, y, z\}| = 1$, then, without loss of generality, $z \in D$ but $D \cup \{y\}$ is a larger 2-independent set of T, a contradiction. Thus $|D \cap \{x, y, z\}| = 2$. Also $D \cap V(T')$ is a 2-independent set of T', implying that $\beta_2(T') \geq \beta_2(T) - 2$. Hence $\beta_2(T) = \beta_2(T') + 2$.

(2) If S' is a $\gamma_2(T')$-set, then $S' \cup \{z, x\}$ is a 2-dominating set of T, and so $\gamma_2(T) \leq \gamma_2(T') + 2$. Assume now that T satisfies $\beta_2(T) = \gamma_2(T)$. If $\gamma_2(T) < \gamma_2(T') + 2$, then by item (1) we have

$$\beta_2(T') + 2 = \beta_2(T) = \gamma_2(T) < \gamma_2(T') + 2,$$

implying that $\beta_2(T') < \gamma_2(T')$, a contradiction. Therefore if $\beta_2(T) = \gamma_2(T)$, then $\gamma_2(T) = \gamma_2(T') + 2$. ■

2.2. Main result

For the purpose of characterizing (γ_2, β_2)-trees, we define the family $\mathcal{O}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ of trees, where T_1 is a star $K_{1,p}$ $(p \geq 1)$, $T = T_k$, and, if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the operations defined below.

- **Operation** $\mathcal{O}_1$: Add a star $K_{1,p}$, $p \geq 2$, centered at a vertex u and join u by an edge to a vertex of T_i.
- **Operation** $\mathcal{O}_2$: Add a double star $S_{1,p}$ with support vertices u and v, where $|L_v| = p$ and join v by an edge to a vertex w of T_i with the condition that if $\gamma_2(T_i - w) = \gamma_2(T_i) - 1$, then no neighbor of w in T_i belongs to a $\gamma_2(T_i - w)$-set.

- **Operation** $\mathcal{O}_3$: Add a path $P_2 = u'u$ and join u by an edge to a leaf v of T_i that belongs to every $\beta_2(T_i)$-set and satisfies in addition $\beta_2(T_i - v) + 1 = \beta_2(T_i)$.
- **Operation** $\mathcal{O}_4$: Add a path $P_3 = u'uv$ and join v by an edge to a vertex w that belongs to a $\gamma_2(T_i)$-set and satisfies further $\gamma_2(T_i - w) \leq \gamma_2(T_i)$, with the condition that if $\gamma_2(T_i - w) = \gamma_2(T_i) - 1$, then no neighbor of w in T_i belongs to a $\gamma_2(T_i - w)$-set.

We state the following lemma.

Lemma 7. *If* $T \in \mathcal{O}$ *then,* $\gamma_2(T) = \beta_2(T)$.

Proof. Let T be a tree of $\mathcal{O}$. Then T is obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is a star $K_{1,p}$ ($p \geq 1$), $T = T_k$, and, if $k \geq 2$, T_{k+1} is obtained recursively from T_k by one of the four operations defined above. We use an induction on the number of operations performed to construct T. Clearly the property is true if $k = 1$. This establishes the basis case.

Assume now that $k \geq 2$ and that the result holds for all trees $T \in \mathcal{O}$ that can be constructed from a sequence of length at most $k - 1$, and let $T' = T_{k-1}$. By the inductive hypothesis, T' is a (γ_2, β_2)-tree. Let T be a tree obtained from T' by using one of the operations $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$. We examine each of the following cases. Note that we will use in the proof the same notation as used for the construction.

Case 1. T is obtained from T' by using operation $\mathcal{O}_1$. By Observation 4, $\gamma_2(T) = \gamma_2(T') + p$ and $\beta_2(T) = \beta_2(T') + p$. Since T' is a (γ_2, β_2)-tree it follows that $\gamma_2(T) = \beta_2(T)$.

Case 2. T is obtained from T' by using operation $\mathcal{O}_2$. By Observation 5, $\beta_2(T) = \beta_2(T') + (p + 2)$ and $\gamma_2(T) \leq \gamma_2(T') + (p + 2)$. Now assume that $\gamma_2(T) < \gamma_2(T') + (p+2)$ and let D be a $\gamma_2(T)$-set. Then, without loss of generality, D contains $L_v \cup \{v\}$ and the unique leaf neighbor of u. If $w \in D$, then $D \cap V(T')$ is a 2-dominating set of T' with cardinality $\gamma_2(T) - (p + 2) < \gamma_2(T')$, which is impossible. Hence $w \notin D$ and so $D' = D \cap V(T')$ is a 2-dominating set of $T' - w$. Note that since $w \notin D$ and $v \in D$, D' contains a neighbor of w in T'. Hence $\gamma_2(T' - w) \leq |D'| = \gamma_2(T) - (p + 2) < \gamma_2(T')$. It follows from Observation 2 that $\gamma_2(T' - w) = \gamma_2(T') - 1$ and D' is a $\gamma_2(T' - w)$-set containing a neighbor of w, a contradiction with the construction. Therefore $\gamma_2(T) = \gamma_2(T') + (p + 2)$. Now using the fact that $\gamma_2(T') = \beta_2(T')$ we obtain $\gamma_2(T) = \beta_2(T)$, that is T is a (γ_2, β_2)-tree.

Case 3. T is obtained from T' by using operation $\mathcal{O}_3$. By Observation 4, $\gamma_2(T') = \gamma_2(T) - 1$. Also $\beta_2(T) \geq \beta_2(T') + 1$ since every $\beta_2(T')$-set can be extended to a 2-independent set of T by adding u'. Now assume that $\beta_2(T) > \beta_2(T') + 1$ and let S be a $\beta_2(T)$-set. Since $\beta_2(T') \geq |S \cap V(T')|$, it follows that

$u, u' \in S$. Hence $v \notin S$ and $S \cap V(T')$ is a 2-independent set of $T' - v$. Thus $\beta_2(T' - v) \geq |S \cap V(T')| = \beta_2(T) - 2$. Also from the construction v satisfies $\beta_2(T' - v) + 1 = \beta_2(T')$. Therefore

$$\beta_2(T') - 1 = \beta_2(T' - v) \geq \beta_2(T) - 2 > \big(\beta_2(T') + 1\big) - 2,$$

a contradiction. Consequently $\beta_2(T) = \beta_2(T') + 1$. Since $\gamma_2(T') = \beta_2(T')$ we obtain $\gamma_2(T) = \beta_2(T)$.

Case 4. T is obtained from T' by using operation $\mathcal{O}_4$. By Observation 6, $\beta_2(T) = \beta_2(T') + 2$ and $\gamma_2(T) \leq \gamma_2(T') + 2$. Assume that $\gamma_2(T) < \gamma_2(T') + 2$ and let D be a $\gamma_2(T)$-set. Clearly $u' \in D$ and $|D \cap \{u', u, v\}| = 2$. If $u \in D$, then $v \notin D$ and so $w \in D$. Hence $D \cap V(T')$ is a 2-dominating set of T' having cardinality $|D| - 2 < \gamma_2(T')$, a contradiction. Therefore $u \notin D$ and so $v \in D$. If $w \in D$, then using the same argument than used above leads to a contradiction. Thus $w \notin D$ and hence $D \cap V(T')$ is a 2-dominating set of $T' - w$. It follows that $\gamma_2(T' - w) \leq |D| - 2 < \gamma_2(T')$ and by Observation 2 we obtain $\gamma_2(T' - w) = \gamma_2(T') - 1$. Therefore $D \cap V(T')$ is a $\gamma_2(T' - w)$-set. Note that w is 2-dominated in T by v and some vertex, say $w' \in V(T')$. But then w' belongs to a $\gamma_2(T' - w)$-set, a contradiction with the construction. Consequently, $\gamma_2(T) = \gamma_2(T') + 2$ implying that $\gamma_2(T) = \beta_2(T)$, that is, T is a (γ_2, β_2)-tree. ■

We now are ready to state our main result.

Theorem 8. *Let T be a tree of order n. Then $\gamma_2(T) = \beta_2(T)$ if and only if $T = K_1$ or $T \in \mathcal{O}$.*

Proof. If $T = K_1$, then $\gamma_2(T) = \beta_2(T)$. If $T \in \mathcal{O}$, then by Lemma 7, $\gamma_2(T) = \beta_2(T)$. Let us prove the necessity. Obviously, $\gamma_2(K_1) = \beta_2(K_1)$, so assume $n \geq 2$. We use an induction on the order n of T. If $n = 2$, then $T = K_{1,1}$ that belongs to $\mathcal{O}$. Assume that every (γ_2, β_2)-tree T' of order $2 \leq n' < n$ is in $\mathcal{O}$. Let T be (γ_2, β_2)-tree of order n. If T is a star, then $T \in \mathcal{O}$. If T is a double star, then T is obtained from T_1 by using Operation $\mathcal{O}_1$ if $n \geq 5$, and T is obtained from $T_1 = K_{1,1}$ by using Operation $\mathcal{O}_3$ if $n = 4$. Therefore both stars and double stars are in $\mathcal{O}$. Thus we may assume that T has diameter at least four.

We now root T at a leaf r of a longest path. Among all vertices at distance $\mathrm{diam}(T) - 1$ from r on a longest path starting at r, let u be one of maximum degree. Since $\mathrm{diam}(T) \geq 4$, let v, w be the parents of u and v, respectively. Also let D be a set that is both $\beta_2(T)$-set and $\gamma_2(T)$-set. Recall that such a set exists as mentioned in the introduction (see [3]). Denote by T_x the subtree induced by a vertex x and its descendants in the rooted tree T. We examine the following cases.

Case 1. $\deg_T(u) \geq 3$, that is u is adjacent to at least two leaves. Let $T' = T - T_u$. By Observation 4, $\gamma_2(T) = \gamma_2(T') + |L_u|$ and $\beta_2(T) = \beta_2(T') + |L_u|$.

Hence $\gamma_2(T') = \beta_2(T')$. By induction on T', $T' \in \mathcal{O}$ and so $T \in \mathcal{O}$ because it is obtained from T' by using operation $\mathcal{O}_1$.

Case 2. $\deg_T(u) = 2$. Let u' be the unique leaf neighbor of u. By our choice of u, every child of v has degree at most two. First we claim that every child of v besides u (if any) is a leaf. Suppose to the contrary that a child b of v is a support vertex with $L_b = \{b'\}$. Then $u', b' \in D$. If $v \in D$, then $u, b \notin D$ (since D is a $\beta_2(T)$-set) but $\{u, b\} \cup D - \{v\}$ would be a 2-independent set of T larger than D, a contradiction. Hence $v \notin D$ and so $u, b \in D$ but $\{v\} \cup D - \{u, b\}$ would be a 2-dominating set of T smaller than D, a contradiction too. Thus every child of v besides u is a leaf. We consider two subcases.

Subcase 2.1. $\deg_T(v) \geq 3$. Hence v is a support vertex and T_v is a double star $S_{1,|L_v|}$. Let $T' = T - T_v$. Clearly T' is nontrivial. By Observation 5, $\gamma_2(T) = \gamma_2(T') + |L_v| + 2$ and $\beta_2(T) = \beta_2(T') + |L_v| + 2$. It follows that $\gamma_2(T') = \beta_2(T')$ and by induction on T', $T' \in \mathcal{O}$. Assume now that $T' - w$ admits a $\gamma_2(T' - w)$-set D'' such that $|D''| = \gamma_2(T') - 1$ and D'' contains at least one vertex adjacent to w in T'. Then $D'' \cup L_v \cup \{u', v\}$ is a 2-dominating set of T', and so

$$\begin{aligned}\gamma_2(T) &\leq \left|D'' \cup L_v \cup \{u', v\}\right| = \gamma_2(T' - w) + |L_v| + 2\\ &= \gamma_2(T') - 1 + |L_v| + 2 < \gamma_2(T') + |L_v| + 2,\end{aligned}$$

a contradiction. Hence such a case cannot occur and so T can be obtained from T' by using operation $\mathcal{O}_2$. Therefore $T \in \mathcal{O}$.

Subcase 2.2. $\deg_T(v) = 2$. Clearly $u' \in D$. Three possibilities can occur ($u \notin D$ and $v, w \in D$), ($u, w \notin D$ and $v \in D$) and ($u, w \in D$ and $v \notin D$). Observe that if the first situation occurs, then $\{u\} \cup D - \{v\}$ is both $\beta_2(T)$-set and $\gamma_2(T)$-set too. Hence we have to consider only the last two situations.

Assume that $u, w \notin D$ and $v \in D$ and let $T' = T - \{u, u'\}$. By Observation 4, $\gamma_2(T') = \gamma_2(T) - 1$. Also it is clear that $\beta_2(T) \geq \beta_2(T') + 1$. If $\beta_2(T) > \beta_2(T') + 1$, then $\gamma_2(T') + 1 = \gamma_2(T) = \beta_2(T) > \beta_2(T') + 1$, implying that $\gamma_2(T') > \beta_2(T')$, a contradiction. Hence $\beta_2(T) = \beta_2(T') + 1$ and so $\gamma_2(T') = \beta_2(T')$. By induction on T', $T' \in \mathcal{O}$. Note that v belongs to every $\beta_2(T')$-set, for otherwise if S' is a $\beta_2(T')$-set such that $v \notin S'$, then $S' \cup \{u, u'\}$ would be a 2-independent set of T larger than D, a contradiction. On the other hand, by Observation 3, $\beta_2(T' - v) \leq \beta_2(T') \leq \beta_2(T' - v) + 1$. Clearly if $\beta_2(T' - v) = \beta_2(T')$, then every $\beta_2(T' - v)$-set is also a $\beta_2(T')$-set but does not contain v, a contradiction with the fact that v belongs to every $\beta_2(T')$-set. Therefore v satisfies $\beta_2(T') = \beta_2(T' - v) + 1$. It follows that $T \in \mathcal{O}$ because it is obtained from T' by using Operation $\mathcal{O}_3$.

Finally assume that $u, w \in D$ and $v \notin D$. Let $T' = T - \{v, u, u'\}$. Then by Observation 6, $\beta_2(T) = \beta_2(T') + 2$ and $\gamma_2(T) = \gamma_2(T') + 2$. Note that $D \cap V(T')$ is a $\gamma_2(T')$-set that contains w. Also by Observation 2, $\gamma_2(T' - w) \geq \gamma_2(T') - 1$.

Assume that $\gamma_2(T'-w) > \gamma_2(T')$. Then using the fact that $\beta_2(T) \geq \beta_2(T'-w)+2$, it follows that

$$\beta_2(T) \geq \beta_2(T'-w)+2 \geq \gamma_2(T'-w)+2 > \gamma_2(T')+2 = \gamma_2(T),$$

and so $\beta_2(T) > \gamma_2(T)$, a contradiction. Therefore $\gamma_2(T') \geq \gamma_2(T'-w) \geq \gamma_2(T')-1$. Now we note that if $\gamma_2(T'-w) = \gamma_2(T')-1$, then no neighbor of w in T' belongs to a $\gamma_2(T'-w)$-set, for otherwise such a set can be extended to 2-dominating set of T by adding u', v which leads to $\beta_2(T) > \gamma_2(T)$. Under these conditions it is clear that T is obtained from T' by using Operation $\mathcal{O}_4$ and since $T' \in \mathcal{O}$ it follows immediately that $T \in \mathcal{O}$. ■

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