

### 3-TRANSITIVE DIGRAPHS

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#### Abstract

Let  $D$  be a digraph,  $V(D)$  and  $A(D)$  will denote the sets of vertices and arcs of  $D$ , respectively.

A digraph  $D$  is 3-transitive if the existence of the directed path  $(u, v, w, x)$  of length 3 in  $D$  implies the existence of the arc  $(u, x) \in A(D)$ . In this article strong 3-transitive digraphs are characterized and the structure of non-strong 3-transitive digraphs is described. The results are used, e.g., to characterize 3-transitive digraphs that are transitive and to characterize 3-transitive digraphs with a kernel.

**Keywords:** digraph, kernel, transitive digraph, quasi-transitive digraph, 3-transitive digraph, 3-quasi-transitive digraph.

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#### 1. INTRODUCTION

In this work,  $D = (V(D), A(D))$  will denote a finite digraph without loops or multiple arcs in the same direction, with vertex set  $V(D)$  and arc set  $A(D)$ . For general concepts and notation we refer the reader to [1, 4] and [7], particularly we will use the notation of [7] for walks, if  $\mathcal{C} = (x_0, x_1, \dots, x_n)$  is a walk and  $i < j$  then  $x_i \mathcal{C} x_j$  will denote the subwalk  $(x_i, x_{i+1}, \dots, x_{j-1}, x_j)$  of  $\mathcal{C}$ . Union of walks will be denoted by concatenation or with  $\cup$ . For a vertex  $v \in V(D)$ , we define the *out-neighborhood* of  $v$  in  $D$  as the set  $N_D^+(v) = \{u \in V(D) \mid (v, u) \in A(D)\}$ ; when there is no possibility of confusion we will omit the subscript  $D$ . The elements of  $N^+(v)$  are called the *out-neighbors* of  $v$ , and the *out-degree* of  $v$ ,  $d_D^+(v)$ , is the number of out-neighbors of  $v$ . Definitions of in-neighborhood, in-neighbors and in-degree of  $v$  are analogously given. We say that a vertex  $u$  reaches a vertex  $v$  in

$D$  if a directed  $uv$ -directed path (a path with initial vertex  $u$  and terminal vertex  $v$ ) exists in  $D$ . An arc  $(u, v) \in A(D)$  is called *asymmetrical* (resp. *symmetrical*) if  $(v, u) \notin A(D)$  (resp.  $(v, u) \in A(D)$ ).

If  $D$  is a digraph and  $X, Y \subseteq V(D)$ , an  $XY$ -arc is an arc with initial vertex in  $X$  and terminal vertex in  $Y$ . If  $X \cap Y = \emptyset$ ,  $X \rightarrow Y$  will denote that  $(x, y) \in A(D)$  for every  $x \in X$  and  $y \in Y$ . Again, if  $X$  and  $Y$  are disjoint,  $X \Rightarrow Y$  will denote that there are not  $YX$ -arcs in  $D$ . When  $X \rightarrow Y$  and  $X \Rightarrow Y$  we will simply write  $X \mapsto Y$ . If  $D_1, D_2$  are subdigraphs of  $D$ , we will abuse notation to write  $D_1 \rightarrow D_2$  or  $D_1 D_2$ -arc, instead of  $V(D_1) \rightarrow V(D_2)$  or  $V(D_1)V(D_2)$ -arc, respectively. Also, if  $X = \{v\}$ , we will abuse notation to write  $v \rightarrow Y$  or  $vY$ -arc instead of  $\{v\} \rightarrow Y$  or  $\{v\}Y$ -arc, respectively. Analogously if  $Y = \{v\}$ .

A digraph is *strongly connected* (or strong) if for every  $u, v \in V(D)$ , there exists a  $uv$ -directed path, i.e., a directed path with initial vertex  $u$  and terminal vertex  $v$ . A *strong component* (or component) of  $D$  is a maximal strong subdigraph of  $D$ . The *condensation* of  $D$  is the digraph  $D^*$  with  $V(D^*)$  equal to the set of all strong components of  $D$ , and  $(S, T) \in A(D^*)$  if and only if there is an  $ST$ -arc in  $D$ . Clearly  $D^*$  is an acyclic digraph (a digraph without directed cycles), and thus, it has both vertices of out-degree equal to zero and vertices of in-degree equal to zero. A *terminal component* of  $D$  is a strong component  $T$  of  $D$  such that  $d_{D^*}^+(T) = 0$ . An *initial component* of  $D$  is a strong component  $S$  of  $D$  such that  $d_{D^*}^-(S) = 0$ .

A *biorientation* of the graph  $G$  is a digraph  $D$  obtained from  $G$  by replacing each edge  $\{x, y\} \in E(G)$  by either the arc  $(x, y)$  or the arc  $(y, x)$  or the pair of arcs  $(x, y)$  and  $(y, x)$ . A *semicomplete* digraph is a biorientation of a complete graph. An *orientation* of a graph  $G$  is an asymmetrical biorientation of  $G$ ; thus, an *oriented* graph is an asymmetrical digraph. A tournament is an orientation of a complete graph. An orientation of a digraph  $D$  is a maximal asymmetrical subdigraph of  $D$ . A *complete* digraph is a biorientation of a complete graph obtained by replacing each edge  $\{x, y\}$  by the arcs  $(x, y)$  and  $(y, x)$ .

Let  $D$  be a digraph with vertex set  $V(D) = \{v_1, v_2, \dots, v_n\}$  and  $H_1, H_2, \dots, H_n$  a family of vertex disjoint digraphs. The *composition* of digraphs  $D[H_1, H_2, \dots, H_n]$  is the digraph having  $\bigcup_{i=1}^n V(H_i)$  as its vertex set and arc set  $\bigcup_{i=1}^n A(H_i) \cup \{(u, v) \mid u \in V(H_i), v \in V(H_j), (v_i, v_j) \in A(D)\}$ . The dual (or converse) of  $D$ ,  $\overleftarrow{D}$  is the digraph with vertex set  $V(\overleftarrow{D}) = V(D)$  and such that  $(u, v) \in A(\overleftarrow{D})$  if and only if  $(v, u) \in A(D)$ . The directed cycle of length 3 will be denoted, as usual, by  $C_3$ .

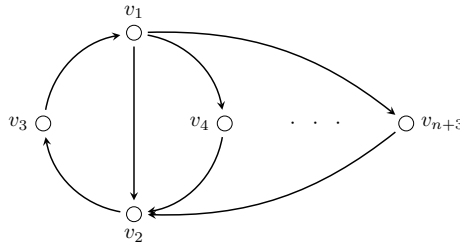
A digraph is *transitive* if for every three distinct vertices  $u, v, w \in V(D)$ ,  $(u, v), (v, w) \in A(D)$  implies that  $(u, w) \in A(D)$ . Transitive digraphs have a lot of properties, many of which can be verified straightforward by using the following structural characterization, which can be found in [1] as an exercise.

**Theorem 1.** *Let  $D$  be a digraph  $D$  with strong components  $S_1, S_2, \dots, S_n$ . Then  $D$  is a transitive digraph if and only if  $D = D^*[S_1, S_2, \dots, S_n]$ , where  $S_i$  is a complete digraph for  $1 \leq i \leq n$ .*

But, the structure of transitive digraphs is so rich that, working on this family, many problems become trivial or have a very simple solution. In view of this situation, some generalizations of transitive digraphs have been studied. Without doubt, the most studied generalization of transitive digraphs is the family of quasi-transitive digraphs. A digraph is quasi-transitive if for every three distinct vertices  $u, v, w \in V(D)$ ,  $(u, v), (v, w) \in A(D)$  implies that  $(u, w) \in A(D)$  or  $(w, u) \in A(D)$ . Clearly, every semicomplete digraph is a quasi-transitive digraph, so, quasi-transitive digraphs generalize both, transitive and semicomplete digraphs. Quasi-transitive have been characterized by Bang Jensen and Huang in [2], and their structure is very similar to the structure of transitive digraphs. Once again, this structural characterization has been very helpful to solve a large number of problems over this family, e.g., characterization of quasi-transitive digraphs with 3-kings, Hamiltonicity in quasi-transitive digraphs, or the Laborde-Payan-Xuong Conjecture for quasi-transitive digraphs.

Quasi-transitive digraphs were generalized with 3-quasi-transitive digraphs. A digraph  $D$  is 3-quasi-transitive if for every directed path,  $(v_0, v_1, v_2, v_3)$ , either  $(v_0, v_3) \in A(D)$  or  $(v_3, v_0) \in A(D)$ . Let us notice that in the definition of 3-quasi-transitive digraphs, the subdigraph  $(v_0, v_1, v_2, v_3)$  considered is a directed path, so it cannot happen that  $v_0 = v_3$  and we can effectively work on digraphs without loops. The family of 3-quasi-transitive digraphs were introduced by Bang-Jensen in the context of arc-locally semicomplete digraphs, which generalize both, semicomplete digraphs and semicomplete bipartite digraphs. A digraph is arc-locally in-semicomplete if  $(z, x), (x, y), (w, y) \in A(D)$  and  $z \neq w$  implies that  $(z, w) \in A(D)$  or  $(w, z) \in A(D)$ . A digraph is arc-locally out-semicomplete if  $(x, z), (x, y), (y, w) \in A(D)$  and  $z \neq w$  implies that  $(x, w) \in A(D)$  or  $(w, x) \in A(D)$ . A digraph is arc-locally semicomplete if it is arc-locally in-semicomplete and arc-locally out-semicomplete. These families are defined to fulfill a property on some specific orientation of a path of length 3, in all of them, the existence of a (undirected) 4-cycle can be inferred from the existence of the specific orientation. There is one more orientation of a directed path of length 3 that induces the existence of a fourth family of digraphs. A digraph is often called of the type  $\mathcal{H}_4$  if  $(x, w), (x, y), (z, y) \in A(D)$  and  $z \neq w$  implies that  $(w, z) \in A(D)$  or  $(z, w) \in A(D)$ . The problem of finding structural characterizations of these four families of digraphs was proposed by Bang-Jensen. Besides transitive and quasi-transitive digraphs, also arc-locally semicomplete digraphs [8] and arc-locally in-semicomplete digraphs [13] have been characterized.

In [10], Galeana-Sánchez and the author introduce  $k$ -transitive and  $k$ -quasi-transitive digraphs. A digraph  $D$  is  $k$ -transitive if the existence of a directed

Figure 1. The family of digraphs  $F_n$ .

path  $(v_0, v_1, \dots, v_k)$  of length  $k$  in  $D$  implies that  $(v_0, v_k) \in A(D)$ . A digraph  $D$  is  $k$ -quasi-transitive if the existence of a directed path  $(v_0, v_1, \dots, v_k)$  of length  $k$  in  $D$  implies that  $(v_0, v_k) \in A(D)$  or  $(v_k, v_0) \in A(D)$ . Also in [10], some basic properties on the structure of both  $k$ -transitive and  $k$ -quasi-transitive are proved. These properties are used to prove the existence of  $n$ -kernels in both families.

The aim of this article is to characterize strong 3-transitive digraphs and give a thorough description of the structure of non-strong 3-transitive digraphs. We will use the following characterization of strong 3-quasi-transitive digraphs given by Galeana-Sánchez, Goldfeder and Urrutia in [9].

**Theorem 2** (Galeana-Sánchez, Goldfeder, Urrutia). *Let  $D$  be a strong 3-quasi-transitive digraph of order  $n$ . Then  $D$  is either a semicomplete digraph, a semicomplete bipartite digraph or isomorphic to  $F_n$  (Figure 1).*

Thus, Section 2 will be devoted to prove some basic results about 3-transitive digraphs. In Section 3 the characterization of strong 3-transitive digraphs and the structural description of non-strong 3-transitive digraphs are given. In Section 4, one application of the results of Section 3 is given: A characterization of 3-transitive digraphs having a kernel. Also, an interesting problem concerning underlying graphs of 3-transitive and 3-quasi-transitive digraphs is proposed.

## 2. PRELIMINARY RESULTS

We begin this section with a very simple remark that will be very useful through this work.

**Remark 3.** A digraph  $D$  is a 3-transitive digraph if and only if  $\overleftarrow{D}$  is 3-transitive.

The following is another simple, yet useful, property of  $k$ -transitive digraphs.

**Proposition 4.** *If  $D$  is a  $k$ -transitive digraph with  $k \geq 2$ , then  $D$  is  $k + n(k-1)$ -transitive for every  $n \in \mathbb{N}$ .*

**Proof.** Let  $D$  be a  $k$ -transitive digraph. We will proceed by induction on  $n$ .

For  $n = 1$ , consider  $(v_0, v_1, \dots, v_{k+(k-1)})$ , a directed path of length  $k+(k-1)$ . From the  $k$ -transitivity of  $D$  we have that  $(v_0, v_k) \in A(D)$ , so  $(v_0, v_k, v_{k+1}, \dots, v_{k+(k-1)})$  is a  $v_0 v_k$ -directed path of length  $k$ , and by the  $k$ -transitivity of  $D$ , we have that  $(v_0, v_{k+(k-1)}) \in A(D)$ .

Let us assume the result valid for  $n - 1$  and let  $(v_0, v_1, \dots, v_{k+n(k-1)})$  be a directed path of length  $k + n(k - 1)$  in  $D$ . By the induction hypothesis  $(v_0, v_{k+(n-1)(k-1)}) \in A(D)$ , and clearly  $(v_0, v_{k+(n-1)(k-1)}, \dots, v_{k+n(k-1)})$  is a directed path of length  $k$  in  $D$ .

It follows from the  $k$ -transitivity that  $(v_0, v_{k+n(k-1)}) \in A(D)$ . The result is now obtained by the Principle of Mathematical Induction. ■

As a particular case of Proposition 4, we can observe that a 3-transitive digraph is  $n$ -transitive for every odd integer  $n$ . We can state this observation as the following corollary.

**Corollary 5.** *Let  $D$  be a 3-transitive digraph and  $(v_0, v_1, \dots, v_n)$  a directed path in  $D$ . Then  $(v_0, v_i) \in A(D)$  for every odd integer  $1 \leq i \leq n$ .*

**Proof.** It is straightforward from Proposition 4. ■

In [14], Wang and Wang proved some results describing the structure of non-strong 3-quasi-transitive digraphs. Since every 3-transitive digraph is also 3-quasi-transitive, the properties stated next hold also for 3-transitive digraphs.

**Proposition 6** [14]. *Let  $D'$  be a non-trivial strong induced subdigraph of a 3-quasi-transitive digraph  $D$  and let  $s \in V(D) \setminus V(D')$  with at least one arc from  $D'$  to  $s$  and  $D' \Rightarrow s$ . Then each of the following holds:*

1. *If  $D'$  is a bipartite digraph with bipartition  $(X, Y)$  and there exists a vertex of  $X$  which dominates  $s$ , then  $X \mapsto s$ .*
2. *If  $D'$  is a non-bipartite digraph, then  $D' \mapsto s$ .*

In the case of 3-transitive digraphs, the condition  $D' \Rightarrow s$  in Proposition 6 not necessary. The following proposition is some kind of analogous of Proposition 6 for 3-transitive digraphs, emphasizing the behavior of certain strong subdigraphs.

**Proposition 7.** *Let  $D$  be a 3-transitive digraph and  $v \in V(D)$ . The following statements hold:*

1. *For every  $C_3$  in  $D$  such that there is a  $C_3 v$ -arc in  $D$ , then  $C_3 \rightarrow v$ .*
2. *For every  $C_3$  in  $D$  such that there is a  $v C_3$ -arc in  $D$ , then  $v \rightarrow C_3$ .*
3. *For every  $\overleftrightarrow{K}_n$  in  $D$ ,  $n \geq 3$ , such that there is a  $\overleftrightarrow{K}_n v$ -arc in  $D$ , then  $\overleftrightarrow{K}_n \rightarrow v$ .*
4. *For every  $\overleftrightarrow{K}_n$  in  $D$ ,  $n \geq 3$ , such that there is a  $v \overleftrightarrow{K}_n$ -arc in  $D$ , then  $v \rightarrow \overleftrightarrow{K}_n$ .*

5. For every  $\overleftrightarrow{K}_{n,m} = (X, Y)$  in  $D$  such that there is a  $Xv$ -arc in  $D$ , then  $X \rightarrow v$ .
6. For every  $\overleftrightarrow{K}_{n,m} = (X, Y)$  in  $D$  such that there is a  $vX$ -arc in  $D$ , then  $v \rightarrow X$ .

**Proof.** For 1. Let  $C_3 = (x, y, z, x)$  be a cycle in  $D$  and  $(x, v) \in A(D)$ . The existence of the directed path  $(y, z, x, v)$  in  $D$ , implies that  $(y, v) \in A(D)$ . Finally, since  $(z, x, y, v)$  is a directed path of length 3 in  $D$ ,  $(z, v) \in A(D)$ . Thus  $C_3 \rightarrow v$ .

For 2. It suffices to dualize 1 using Remark 3.

For 3. Let  $D[S]$ , with  $S = \{1, 2, \dots, n\}$ , be a complete subdigraph of  $D$  and  $(1, v) \in A(D)$ . Let  $i \in S \setminus \{1\}$  be an arbitrary vertex. Remember that  $n \geq 3$ , so there exists a vertex  $j \in S \setminus \{1, i\}$ . Now, since  $D[S] = \overleftrightarrow{K}_n$ , we have the existence of the directed path  $(i, j, 1, v)$ , which implies that  $(i, v) \in A(D)$ . But  $i$  is an arbitrary vertex of  $D[S]$ , and then we can conclude that  $D[S] \rightarrow v$ .

For 4. It suffices to dualize 3 using Remark 3.

For 5. Let  $\overleftrightarrow{K}_{n,m} = (X, Y)$  be a complete subdigraph of  $D$  and  $x \in X$ . If  $|X| = 1$ , then we are done. If not, let  $z \in X$  be a vertex such that  $z \neq x$ . Since  $Y \neq \emptyset$ , there is a vertex  $y \in Y$ . Also,  $(z, y), (y, x) \in A(D)$ , because  $D[X \cup Y]$  is a complete bipartite digraph. So  $(z, y, x, v)$  is a directed path of length 3 in  $D$  and hence,  $(z, v) \in A(D)$ . Thus,  $X \rightarrow v$ .

For 6. It suffices to dualize 5 using Remark 3. ■

The following proposition is also due to Wang and Wang.

**Proposition 8** [14]. *Let  $D'$  be a non-trivial strong subdigraph of a 3-quasi-transitive digraph  $D$ . For any  $s \in V(D) \setminus V(D')$ , if there exists a directed path between  $s$  and  $D'$ , then  $s$  and  $D'$  are adjacent.*

In the case of 3-transitive digraphs we can be a little more specific. The proof of the following proposition will be omitted since it is almost the same as the one given by Wang and Wang in [14].

**Proposition 9.** *Let  $D'$  be a non-trivial strong subdigraph of a 3-transitive digraph  $D$  and  $s \in V(D) \setminus V(D')$ . Then each of the following holds:*

1. *If there exists an  $sD'$ -directed path in  $D$ , then an  $sD'$ -arc exists.*
2. *If there exists a  $D's$ -directed path in  $D$ , then a  $D's$ -arc exists.*

The following couple of propositions will be used later to characterize strong 3-transitive digraphs.

**Proposition 10.** *Let  $D$  be a strong 3-transitive digraph of order  $n \geq 4$ . If  $D$  is semicomplete, then  $D$  is complete.*

**Proof.** For any  $(x, y) \in A(D)$ , let  $P = (y_0, y_1, \dots, y_s)$  be a shortest path from  $y$  to  $x$ . If  $s \geq 3$ , then by Corollary 5 we can find a shorter path than  $P$  from  $y$  to

$x$ . Suppose that  $s = 2$ , then  $(x, y, y_1, x)$  is a 3-cycle in  $D$ . Let  $D' = D[\{x, y, y_1\}]$ . Since the order of  $D$  is  $n \geq 4$ , there exists  $v \in V(D) \setminus V(D')$ . Also,  $D$  is strong, so a  $D'$ -directed path and an  $sD'$ -directed path exist in  $D$ . It follows from Propositions 7 (1 and 2) and 9 that  $(y_1, v), (v, x) \in A(D)$ . So  $(y, y_1, v, x)$  is a directed path of length 3 in  $D$  and hence,  $(y, x) \in A(D)$ . This contradicts that  $s = 2$ . Thus,  $(y, x) \in A(D)$ . ■

**Proposition 11.** *Let  $D$  be a strong 3-transitive digraph. If  $D$  is semicomplete bipartite, then  $D$  is complete bipartite.*

**Proof.** Let  $(X, Y)$  be the bipartition of  $D$ . It suffices to prove that for any  $(v, u) \in A(D), (u, v) \in A(D)$ . Since  $D$  is strong, there exists a path  $P$  from  $u$  to  $v$  of length  $n$ . Again, since  $D$  is bipartite and  $u$  and  $v$  belong to the different partite,  $n$  must be odd. By Corollary 5,  $(u, v) \in A(D)$ . ■

### 3. THE STRUCTURE OF 3-TRANSITIVE DIGRAPHS

Let  $C_3^*$  and  $C_3^{**}$  be directed triangles with one and two symmetrical arcs, respectively. Digraphs  $C_3, C_3^*$  and  $C_3^{**}$  are shown in Figure 2.

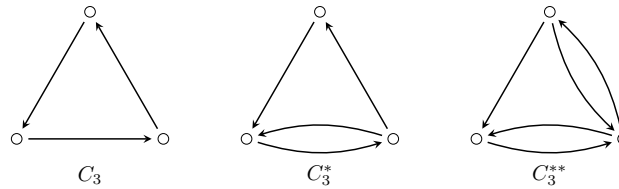


Figure 2. The digraphs  $C_3, C_3^*$  and  $C_3^{**}$ .

The characterization of strong 3-transitive digraphs is now proved.

**Proposition 12.** *A strong digraph  $D$  of order  $n$  is 3-transitive if and only if it is one of the following:*

1. A complete digraph,
2. A complete bipartite digraph,
3.  $C_3, C_3^*$  or  $C_3^{**}$ .

**Proof.** Since every 3-transitive digraph is 3-quasi-transitive, in virtue of Theorem 2, a strong 3-transitive digraph must be either semicomplete, semicomplete bipartite or isomorphic to  $F_n$ . But  $F_n$  is not 3-transitive, so a strong 3-transitive digraph must be either semicomplete or semicomplete bipartite. It is clear that every strong digraph of order less than or equal to 3 is either complete, complete

bipartite or one of the digraphs  $C_3, C_3^*$  or  $C_3^{**}$ . If  $D$  has order greater than or equal to 4, and it is a semicomplete digraph, it follows from Proposition 10 that  $D$  is complete. Finally, if  $D$  is semicomplete bipartite, it follows from Proposition 11 that  $D$  is complete bipartite. ■

As immediate corollary from Proposition 12, we get the following result.

**Corollary 13.** *Let  $D$  be a 3-transitive digraph. Then  $D$  is Hamiltonian if and only if  $D$  is strong and it is not bipartite or it is regular.*

Let us recall that Proposition 7 describes the interaction of a single vertex with some subdigraphs of a 3-transitive digraph  $D$ . This covers the case when a strong component of  $D$  consists of a single vertex. In [14], the following proposition is proved.

**Proposition 14.** *Let  $D_1$  and  $D_2$  be two distinct non-trivial strong components of a 3-quasi-transitive digraph with at least one  $D_1D_2$ -arc. Then either  $D_1 \mapsto D_2$  or the digraph induced by  $D_1 \cup D_2$  is a semicomplete bipartite digraph.*

As it was noted before, every 3-transitive digraph is a 3-quasi-transitive digraph, so Proposition 14 is also valid for 3-transitive digraphs. In an attempt to be more explicit with the interaction between non-trivial strong components of a 3-transitive digraph, we state the following proposition. Nonetheless, we omit the proof, since it is very similar to the proof of Proposition 14.

**Proposition 15.** *Let  $D$  be a 3-transitive digraph and  $S_1, S_2$  be distinct strong components of  $D$  such that there exists an  $S_1S_2$ -arc. The following statements hold:*

1. *If  $S_1$  contains a subdigraph isomorphic to  $C_3$ , then  $S_1 \rightarrow S_2$ .*
2. *If  $S_2$  contains a subdigraph isomorphic to  $C_3$ , then  $S_1 \rightarrow S_2$ .*
3. *If  $S_i$  is a complete bipartite digraph with bipartition  $(X_i, Y_i)$  for  $i \in \{1, 2\}$  and if the  $S_1S_2$ -arc is an  $X_1X_2$ -arc, then  $X_1 \rightarrow X_2$ .*
4. *If  $S_i$  is a complete bipartite digraph with bipartition  $(X_i, Y_i)$  for  $i \in \{1, 2\}$  and there exist an  $X_1X_2$ -arc and a  $Y_1X_2$ -arc, then  $S_1 \rightarrow S_2$ .*
5. *If  $S_i$  is a complete bipartite digraph with bipartition  $(X_i, Y_i)$  for  $i \in \{1, 2\}$  and there exist an  $X_1X_2$ -arc and an  $X_1Y_2$ -arc, then  $S_1 \rightarrow S_2$ .*

As a direct consequence of Propositions 9 and 15, we have the following corollary.

**Corollary 16.** *Let  $D$  be a 3-transitive digraph and  $S_1$  a strong component of  $D$  which contains a subdigraph isomorphic to  $C_3$ . If  $S_1 \rightarrow v$  for some vertex  $v \in V$ , then  $S_1 \rightarrow u$  for every vertex  $u \in V$  that can be reached from  $v$ . Dually, if  $v \rightarrow S_1$  for some vertex  $v \in V$ , then  $u \rightarrow S_1$  for every vertex  $u \in V$  that reaches  $v$ .*



We have already proved that the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. The following results are devoted to a deeper exploration of the similarities between these families of digraphs. A structural characterization of 3-transitive digraphs that are transitive is given.

**Theorem 17.** *Let  $D$  be a non-strong 3-transitive digraph with strong components  $S_1, S_2, \dots, S_p$ . Then  $D = D^*[S_1, S_2, \dots, S_p]$  if and only if, for every pair of strong components  $S_i, S_j$  of  $D$ , such that an  $S_i S_j$ -arc exists in  $D$ , then:*

1. *If  $S_i, S_j$  are complete bipartite digraphs, then  $D[S_i \cup S_j]$  is not bipartite.*
2. *If one of  $S_i$  and  $S_j$  is a complete bipartite digraph and the other consists of a single vertex, then  $D[S_i \cup S_j]$  is not bipartite.*

**Proof.** The necessity is trivial. In order to prove the sufficient, let  $S_i$  and  $S_j$  be two distinct strong components of  $D$  such that there is an  $S_i S_j$ -arc. If both  $S_i$  and  $S_j$  are both non-trivial digraphs, then by 1 of the theorem and Proposition 14, we have that  $S_i \rightarrow S_j$ . Since the converse of a 3-transitive digraph is still a 3-transitive digraph, we assume, without loss of generality, that  $S_i$  is a non-trivial complete bipartite digraph with bipartition  $(X_i, Y_i)$  and  $S_j = \{v\}$ . Since  $D[S_i \cup S_j]$  is not a bipartite digraph, then there is a vertex  $x \in X_i$  such that  $x \rightarrow v$  and there is a vertex  $y \in Y_i$  such that  $y \rightarrow v$ . By Proposition 6.1, we have that  $S_i \rightarrow v$ . ■

**Theorem 18.** *Let  $D$  be a 3-transitive digraph. Then  $D^*$  is a transitive digraph if and only if for every triplet of strong components  $S_1, S_2, S_3$  of  $D$ , such that:  $S_i$  consists of a single vertex  $v_i$ ,  $i \in \{1, 3\}$ ;  $S_2$  is either a single vertex  $v_2$  or a complete bipartite digraph with bipartition  $(X, Y)$  and  $v_1 \rightarrow v_2 \rightarrow v_3$  or  $v_1 \rightarrow X \rightarrow v_3$  but there are neither  $v_1 Y$ -arcs nor  $Y v_3$ -arcs in  $D$ , respectively, then  $(v_1, v_3) \in A(D)$ .*

**Proof.** Let  $D$  be a 3-transitive digraph. If  $D^*$  is a transitive digraph, then for every triplet of strong components  $S_1, S_2$  and  $S_3$  of  $D$ , such that there is an  $S_1 S_2$ -arc in  $D$  and an  $S_2 S_3$ -arc in  $D$ , then there is an  $S_1 S_3$ -arc in  $D$ . In particular, if  $S_1$  and  $S_3$  consist of single vertices  $v_1$  and  $v_3$  respectively, then  $(v_1, v_3) \in A(D)$ .

For the converse, let  $D$  be a 3-transitive digraph and  $S_1, S_2$  and  $S_3$  strong components of  $D$ , such that there is an  $S_1 S_2$ -arc in  $D$  and an  $S_2 S_3$ -arc in  $D$ . We will prove that there is an  $S_1 S_3$ -arc in  $D$ . If  $S_1$  contains an isomorphic copy of  $C_3$ , then, by Corollary 16, we have that  $S_1 \rightarrow S_3$  in  $D$ . If  $S_3$  contains an isomorphic copy of  $C_3$ , again, by Corollary 16, we have that  $S_1 \rightarrow S_3$ . So, let us assume that neither  $S_1$  nor  $S_3$  contains an isomorphic copy of  $C_3$ .

It follows from Proposition 12 that  $S_1$  and  $S_3$  are either a single vertex or complete bipartite digraphs. If  $S_1$  is not a single vertex, then it is a complete bipartite digraph with bipartition  $(X_1, Y_1)$ . Let us assume without loss of generality that the  $S_1 S_2$ -arc is an  $X_1 S_2$ -arc. Let  $(x_1, u)$  be the  $S_1 S_2$ -arc in  $D$ . Since

$S_2$  is a strong component of  $D$ , we have, by Propositions 12 and 15, two cases. The first case is that a vertex  $s_3 \in V(S_3)$  exists, such that  $(u, s_3) \in A(D)$ . In this case is clear that, for any vertex  $y_1 \in Y_1$  (recall that  $Y_1 \neq \emptyset$ ),  $(y_1, x_1, u, s_3)$  is a directed path of length 3 in  $D$ . By the 3-transitivity of  $D$ , we have that  $(y_1, s_3) \in A(D)$ , the desired  $S_1S_3$ -arc. The second case is that vertices  $v \in V(S_2)$  and  $s_3 \in V(S_3)$  exist, such that  $(u, v), (v, s_3) \in A(D)$ . Again, it is clear that  $(x_1, u, v, s_3)$  is a directed path of length 3 and thus,  $(x_1, s_3) \in A(D)$ , the desired  $S_1S_3$ -arc. The case when  $S_3$  is a complete bipartite digraph can be obtained dualizing the previous argument using Remark 3.

So, the remaining cases are when  $S_1$  and  $S_3$  consist of single vertices. We have again two cases. First, when  $S_2$  contains a subdigraph isomorphic to  $C_3$ , then  $S_2 \rightarrow S_3$ . So, there exist vertices  $s_1 \in V(S_1), u, v \in V(S_2), s_3 \in V(S_3)$  such that  $(s_1, u), (u, v), (v, s_3) \in A(D)$ . Thus,  $(s_1, u, v, s_3)$  is a directed path of length 3 in  $D$ . By the 3-transitivity of  $D$ ,  $(s_1, s_3) \in A(D)$  is the desired  $S_1S_3$ -arc. If  $S_2$  does not contain a subdigraph isomorphic to  $C_3$ , then  $S_2$  is a single vertex or complete bipartite. If  $S_2$  is a single vertex  $v_2$  or a complete bipartite digraph with bipartition  $(X, Y)$  such that  $v_1 \rightarrow v_2 \rightarrow v_3$  or  $v_1 \rightarrow X \rightarrow v_3$  but there are neither  $v_1Y$ -arcs nor  $Yv_3$ -arcs in  $D$ , respectively, then, by hypothesis  $(v_1, v_3) \in A(D)$ . Hence, we have the existence of an  $S_1S_3$ -arc. The remaining case is that  $S_2$  is a complete bipartite digraph with bipartition  $(X, Y)$  such that  $v_1 \rightarrow X \rightarrow v_3$ , and either a  $v_1Y$ -arc or a  $Yv_3$ -arc exists. In the first case we have by Proposition 15 that  $v_1 \rightarrow S_2$ , and thus, vertices  $u \in X, v \in Y$  exist such that  $(v_1, v), (u, v_3) \in A(D)$ . So,  $(v_1, v, u, v_3)$  is a directed path of length 3 in  $D$ . For the second case, again by Proposition 15, it follows that  $S_2 \rightarrow v_3$ . Then, vertices  $u \in X$  and  $v \in Y$  exist such that  $(v_1, u), (v, v_3) \in A(D)$ . Therefore,  $(v_1, u, v, v_3)$  is a directed path of length 3 in  $D$ . In either case, it follows by the 3-transitivity of  $D$  that  $(v_1, v_3) \in A(D)$ . So an  $S_1S_3$ -arc exists.

Since the cases are exhaustive, we have that  $D^*$  is transitive. ■

**Corollary 19.** *Let  $D$  be a 3-transitive digraph. Then  $D$  is a transitive digraph if and only if every strong component of  $D$  is a complete digraph and, for every triplet of strong components  $S_1, S_2, S_3$  of  $D$ , such that:  $S_i$  consists of a single vertex  $v_i, i \in \{1, 3\}$ ;  $S_2$  is either a single vertex  $v_2$  or a symmetrical arc  $(v_2, v'_2) \in A(D)$  and  $v_1 \rightarrow v_2 \rightarrow v_3$  but  $(v_1, v'_2), (v'_2, v_3) \notin A(D)$ , then  $(v_1, v_3) \in A(D)$ .*

**Proof.** It is clear from Theorems 1, 17 and 18. ■

**Corollary 20.** *Let  $D$  be a 3-transitive digraph. If every strong component of  $D$  is a complete digraph of order greater than or equal to 3, then  $D$  is transitive.*

**Proof.** Let  $D$  be a 3-transitive digraph such that every strong component of  $D$  is a complete digraph of order greater than or equal to 3. Then, by Theorem 18, it is clear that  $D^*$  is transitive. Also, in virtue of Theorem 15, we can observe

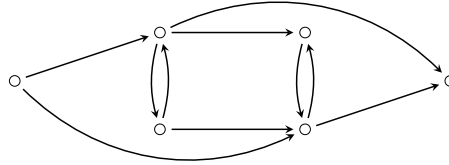


Figure 3. A 3-transitive digraph without 3-transitive condensation.

that  $S_i \rightarrow S_j$  for every pair of strong components  $S_i, S_j$  of  $D$  such that there exists an  $S_i S_j$ -arc in  $D$ . Thus,  $D = D^*[S_1, S_2, \dots, S_n]$ , where  $\{S_1, S_2, \dots, S_n\}$  is the set of strong components of  $D$  and  $D^*$  is transitive. So, by Theorem 1,  $D$  is transitive. ■

As we have already shown, the structure of 3-transitive digraphs is very similar to the structure of transitive digraphs. We know that the condensation of a transitive digraph is again transitive. A characterization of 3-transitive digraphs with a transitive condensation has been already given, but a natural question arises. Is the condensation of a 3-transitive digraph 3-transitive again? Sadly, the answer is no, Figure 3 shows a counterexample to this fact.

Following similar ideas to those used to characterize the 3-transitive digraphs with a transitive condensation in Theorem 18, we can characterize 3-transitive digraphs with a 3-transitive condensation. The ‘bad’ configurations, preventing the condensation of a 3-transitive digraph to be 3-transitive, are pointed out in the following theorem.

**Theorem 21.** *Let  $D$  be a 3-transitive digraph. Then  $D^*$  is a 3-transitive digraph if and only if for every 4-set,  $\{S_1, S_2, S_3, S_4\}$ , of strong components of  $D$  such that:  $S_i$  consists of a single vertex  $v_i$ ,  $i \in \{1, 4\}$  and one of the following conditions is fulfilled:*

1.  $S_2$  consists of single vertex  $v_2$  and  $S_3$  is a complete bipartite digraph with bipartition  $(X, Y)$ , such that  $v_1 \rightarrow v_2 \rightarrow X$  and  $Y \rightarrow v_4$ , but there are neither  $v_2 Y$ -arcs nor  $X v_4$ -arcs in  $D$ ;
2.  $S_2$  is a complete bipartite digraph with bipartition  $(X, Y)$  and  $S_3$  consists of single vertex  $v_3$ , such that  $v_1 \rightarrow X$  and  $Y \rightarrow v_3 \rightarrow v_4$ , but there are neither  $v_1 Y$ -arcs nor  $X v_3$ -arcs in  $D$ ;
3.  $S_j$  is a complete bipartite digraph with bipartition  $(X_j, Y_j)$ ,  $j \in \{2, 3\}$ , such that  $v_1 \rightarrow X_2 \rightarrow X_3$  and  $Y_3 \rightarrow v_4$ , but there are neither  $v_1 Y_2$ -arcs,  $v_1 X_3$ -arcs,  $Y_2 v_4$ -arcs, nor  $X_3 v_4$ -arcs, and  $D[V(S_2) \cup V(S_3)]$  is a semicomplete bipartite digraph,

then  $(v_1, v_4) \in A(D)$ .

**Proof.** Let  $D$  be a 3-transitive digraph. If  $D^*$  is a 3-transitive digraph, then for every 4-set of strong components  $\{S_1, S_2, S_3, S_4\}$  of  $D$ , such that there is an  $S_i S_{i+1}$ -arc in  $D$ ,  $i \in \{1, 2, 3\}$ , then there is an  $S_1 S_3$ -arc in  $D$ . In particular, if  $S_1$  and  $S_4$  consist of single vertices  $v_1$  and  $v_4$  respectively, then  $(v_1, v_4) \in A(D)$ .

Conversely, let  $\{S_1, S_2, S_3, S_4\}$  be a 4-set of strong components of  $D$  such that there is an  $S_i S_{i+1}$ -arc in  $D$ ,  $i \in \{1, 2, 3\}$ . If  $S_1$  or  $S_4$  are non-trivial, then by Proposition 9, there exists an  $S_1 S_4$ -arc in  $D$ . So let us assume without loss of generality that  $S_i$  consists of a single vertex  $S_i$ ,  $i \in \{1, 4\}$ . Suppose that  $S_2$  or  $S_3$  contains  $C_3$  as a subdigraph. It can be easily derived from Corollary 16 the existence of an  $S_1 S_4$ -arc in  $D$ . So, we have 3 cases.

Before the analysis of the cases, let us recall that, by Proposition 7, if  $S = (X, Y)$  is a bipartite strong component of  $D$  and  $v \in V(D) \setminus V(S)$  such that a  $vX$ -arc exists, then  $v \rightarrow X$ ; and if an  $Xv$ -arc exists, then  $X \rightarrow v$ .

The first case is when  $S_2$  consists of single vertex  $v_2$  and  $S_3$  is a complete bipartite digraph with bipartition  $(X, Y)$ . Clearly, if a  $v_2 X$ -arc, and an  $X v_4$ -arc exist, then  $v_2 \rightarrow X \rightarrow v_4$ . Thus, a  $v_1 v_4$ -directed path of length 3 exists and  $(v_1, v_4) \in A(D)$  by the 3-transitivity of  $D$ . Analogously, if a  $v_2 Y$ -arc and a  $Y v_4$ -arc exist in  $D$ , clearly  $(v_1, v_4) \in A(D)$ . So, we can assume without loss of generality that  $v_2 \rightarrow X, Y \rightarrow v_4$  and there are neither  $v_2 Y$ -arcs nor  $X v_4$ -arcs in  $D$ . Then, by hypothesis,  $(v_1, v_4) \in A(D)$ .

The second case is when  $S_2$  is a complete bipartite digraph with bipartition  $(X, Y)$  and  $S_3$  consists of single vertex  $v_3$ . But this case is just the dual of the first case, so, using Remark 3, it can be easily shown that  $(v_1, v_4) \in A(D)$ .

The third case is when  $S_j$  is a complete bipartite digraph with bipartition  $(X_j, Y_j)$ ,  $j \in \{2, 3\}$ . Let us assume without loss of generality that  $v_1 \rightarrow X_2$  and  $Y_3 \rightarrow v_4$ . If  $X_2 \rightarrow Y_3$ , then  $v_1 \rightarrow X_2 \rightarrow Y_3 \rightarrow v_4$  and clearly  $(v_1, v_4) \in A(D)$ . If  $Y_2 \rightarrow X_3$ , it is easy to observe that  $X_2 \rightarrow Y_3$ . So, we can suppose that  $X_2 \rightarrow X_3$  (thus  $Y_2 \rightarrow Y_3$ ) and that there are neither  $X_2 Y_3$ -arcs nor  $Y_2 X_3$ -arcs. Thus,  $D[V(S_2) \cup V(S_3)]$  is semicomplete bipartite. If  $v_1 \rightarrow Y_2$ , then  $v_1 \rightarrow Y_2 \rightarrow Y_3 \rightarrow v_4$  and we are done. If  $v_1 \rightarrow X_3$ , then  $v_1 \rightarrow X_3 \rightarrow Y_3 \rightarrow v_4$  and  $(v_1, v_4) \in A(D)$ . Symmetrically, if  $Y_2 \rightarrow v_4$  or  $X_3 \rightarrow v_4$  we can conclude that  $(v_1, v_4) \in A(D)$ . Hence, we can suppose that there are neither  $v_1 Y_2$ -arcs,  $v_1 X_3$ -arcs,  $Y_2 v_4$ -arcs, nor  $X_3 v_4$ -arcs in  $D$ . By hypothesis  $(v_1, v_4) \in A(D)$ .

Since the cases are exhaustive, we have that  $D^*$  is 3-transitive. ■

## 4. CONSEQUENCES

### 4.1. Existence of kernels

Let  $D$  be a digraph and  $N \subseteq V(D)$ . We say that  $N$  is  $l$ -absorbent if for every vertex  $u \in V(D) \setminus N$ , there is a vertex  $v \in N$  such that  $d(u, v) \leq l$  in  $D$ . The set

$N$  is  $k$ -independent if for every  $u, v \in N$ , we have that  $d(u, v), d(v, u) \geq N$ . We call  $N$  a  $(k, l)$ -kernel of  $D$  if  $D$  is  $k$ -independent and  $l$ -absorbent. A  $(k, k - 1)$ -kernel is a  $k$ -kernel and a 2-kernel is simply a kernel. In [12], von Neumann and Morgenstern introduce the concept of kernel of a digraph in the context of Game Theory. Since then, kernels have been largely studied for their applications within many branches of Mathematics, we can find in [5] a very good survey on the subject. Also, in [6] is proved that the problem of determining if a given digraph has a kernel is  $NP$ -complete, so, finding sufficient conditions for a digraph to have a kernel or finding large families of digraphs with a kernel is a very valuable progress.

**Theorem 22.** *Let  $D$  be a 3-transitive digraph. Then  $D$  has a kernel if and only if it has no terminal strong component isomorphic to  $C_3$ .*

**Proof.** The ‘only if’ part will be proved by contrapositive. Let  $D$  be a 3-transitive digraph such that a terminal strong component  $S$  is isomorphic to  $C_3$ . Let  $V(S) = \{v_0, v_1, v_2\}$  and  $A(S) = \{(v_i, v_{i+1})\}_{i=0}^2 \pmod{3}$ . Since  $S$  is terminal, we have that  $d^+(v) = 1$  for every  $v \in V(S)$ . Thus, the only out-neighbor of  $v_i$  is  $v_{i+1} \pmod{3}$ . It is clear that  $S$  has no kernel and vertices in  $S$  cannot be absorbed by any other vertex in  $D$ , thus,  $D$  has no kernel.

The ‘if’ implication will be proved by induction on the number of strong components of  $D$ . Let us assume that  $D$  is strong. It can be directly verified that the digraphs mentioned in Proposition 12, except for  $C_3$  have a kernel. So, let us assume that every 3-transitive digraph such that no terminal strong component isomorphic to  $C_3$  and with  $n$  strong components has a kernel. Let  $D$  be a 3-transitive digraph such that no terminal strong component isomorphic to  $C_3$  and with  $n + 1$  strong components. Let us recall that  $D^*$  is an acyclic digraph, so, we can consider an initial strong component  $S$  of  $D$ . By induction hypothesis,  $D - S$  has a kernel  $N$ . If  $S$  is not a complete bipartite digraph, then, either  $S$  consists of a single vertex or contains a subdigraph isomorphic to  $C_3$ . If  $S$  consists of a single vertex  $v$ , and  $v$  is absorbed by  $N$ , we are done. If  $v$  is not absorbed by  $N$ , since  $S$  is initial,  $N \cup \{v\}$  is independent and thus a kernel of  $D$ . If  $D$  contains a subdigraph isomorphic to  $C_3$ , we can use Corollary 16 to prove that  $S \mapsto S_t$  for some terminal strong component  $S_t$  of  $D$ . But since  $S_t$  is terminal, at least one vertex of  $S_t$  must belong to  $N$ , and thus  $S$  is absorbed by  $N$ . So,  $N$  is a kernel of  $D$ . If  $S$  is a complete bipartite digraph, we must consider three cases. Let  $(X, Y)$  be the bipartition of  $S$ . If neither  $X$  nor  $Y$  is absorbed by  $N$ , then we consider  $N \cup X$ . Since  $S$  is an initial component, every arc between  $X$  and  $N$  must be an  $XN$ -arc. But if such arc exists, we would have by Proposition 7.5 that  $X \rightarrow N$ , contradicting our assumption. So  $N \cup X$  is an independent set, and  $Y \rightarrow X$  because  $S$  is a complete bipartite digraph. Thus,  $N \cup X$  is a kernel for  $D$ . If some vertex of  $X$  is absorbed by  $N$ , then by Proposition 7.5  $X$  is absorbed

by  $N$ . So let us assume that  $Y$  is not absorbed by  $N$ . Once again, since  $S$  is an initial component, every arc between  $N$  and  $Y$  must be a  $YN$ -arc, but no such arc can exist. So,  $N \cup Y$  is an independent absorbent set of  $D$ , and hence a kernel of  $D$ . The case when  $Y$  is absorbed but  $X$  is not is analogous. Finally, if  $S$  is absorbed by  $N$ , we have that  $N$  is the desired kernel of  $D$ .

Since in every case  $D$  has a kernel, the result follows from the Principle of Mathematical Induction. ■

In [10], Galeana-Sánchez and the author proved that a  $k$ -transitive digraph  $D$  has a  $n$ -kernel for every  $n \geq k$ . Thus, Theorem 22 completes the study of existence of  $k$ -kernels in 3-transitive digraphs.

#### 4.2. One further problem

A graph  $G$  is a comparability graph if it can be oriented as an asymmetrical transitive digraph. In [11], Ghouila-Houri proved that the underlying graphs of asymmetrical quasi-transitive digraphs are comparability graphs. That is to say, a graph  $G$  can receive a transitive orientation if and only if  $G$  can receive a quasi-transitive orientation. In view of this result, and considering the great similarity between the structure of transitive and 3-transitive digraphs, we propose the following conjecture.

**Conjecture 23.** *Let  $D$  be an asymmetrical 3-quasi-transitive digraph, then the underlying graph of  $D$ ,  $UG(D)$ , admit a 3-transitive asymmetrical orientation.*

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