

**BOUNDS ON THE GLOBAL OFFENSIVE  
 $k$ -ALLIANCE NUMBER IN GRAPHS**

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**Abstract**

Let  $G = (V(G), E(G))$  be a graph, and let  $k \geq 1$  be an integer. A set  $S \subseteq V(G)$  is called a *global offensive  $k$ -alliance* if  $|N(v) \cap S| \geq |N(v) - S| + k$  for every  $v \in V(G) - S$ , where  $N(v)$  is the neighborhood of  $v$ . The global offensive  $k$ -alliance number  $\gamma_o^k(G)$  is the minimum cardinality of a global offensive  $k$ -alliance in  $G$ . We present different bounds on  $\gamma_o^k(G)$  in terms of order, maximum degree, independence number, chromatic number and minimum degree.

**Keywords:** global offensive  $k$ -alliance number, independence number, chromatic number.

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## 1. TERMINOLOGY

Let  $G = (V, E) = (V(G), E(G))$  be a finite and simple graph. The *open neighborhood* of a vertex  $v \in V$  is  $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$  and the *closed neighborhood* is  $N_G[v] = N[v] = N(v) \cup \{v\}$ . The *degree* of  $v$ , denoted by  $d_G(v)$ , is  $|N(v)|$ . By  $n(G) = n$ ,  $\Delta(G) = \Delta$  and  $\delta(G) = \delta$  we denote the *order*, the *maximum degree* and the *minimum degree* of the graph  $G$ , respectively. If  $A \subseteq V(G)$ , then  $G[A]$  is the graph induced by the vertex set  $A$ . We denote by  $K_n$  the *complete graph* of order  $n$ , and by  $K_{r,s}$  the *complete bipartite graph* with partite sets  $X$  and  $Y$  such that  $|X| = r$  and  $|Y| = s$ . A set  $D \subseteq V(G)$  is a *k-dominating set* of  $G$  if every vertex of  $V(G) - D$  has at least  $k \geq 1$  neighbors in  $D$ . The *k-domination number*  $\gamma_k(G)$  is the cardinality of a minimum  $k$ -dominating set. The case  $k = 1$  leads to the classical *domination number*  $\gamma(G) = \gamma_1(G)$ .

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive  $k$ -alliances, given by Shafique and Dutton [14, 15]. A set  $S$  of vertices of a graph  $G$  is called a *global offensive k-alliance* if  $|N(v) \cap S| \geq |N(v) - S| + k$  for every  $v \in V(G) - S$ , where  $k \geq 1$  is an integer. The *global offensive k-alliance number*  $\gamma_o^k(G)$  is the minimum cardinality of a global offensive  $k$ -alliance in  $G$ . If  $S$  is a global  $k$ -offensive alliance of  $G$  and  $|S| = \gamma_o^k(G)$ , then we say that  $S$  is a  $\gamma_o^k(G)$ -set. A global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive  $k$ -alliances is *NP*-complete.

If  $k \geq 1$  is an integer, then let  $L_k(G) = \{x \in V(G) : d_G(x) \leq k - 1\}$ . Denote by  $\alpha(G)$  the *independence number*, by  $\chi(G)$  the *chromatic number*, and by  $\omega(G)$  the *clique number* of  $G$ , respectively. The *corona graph*  $G \circ K_1$  of a graph  $G$  is the graph constructed from a copy of  $G$ , where for each vertex  $v \in V(G)$ , a new vertex  $v'$  and a pendant edge  $vv'$  are added. Next assume that  $G_1$  and  $G_2$  are two graphs with disjoint vertex sets. The *union*  $G = G_1 \cup G_2$  has  $V(G) = V(G_1) \cup V(G_2)$  and  $E(G) = E(G_1) \cup E(G_2)$ . The *join*  $G = G_1 + G_2$  has  $V(G) = V(G_1) \cup V(G_2)$  and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

## 2. UPPER BOUNDS

We begin this section by giving an upper bound on the global offensive  $k$ -alliance number for an  $r$ -partite graph  $G$  in terms of its order and  $|L_k(G)|$ .

**Theorem 1.** *Let  $k \geq 1$  be an integer. If  $G$  is an  $r$ -partite graph, then*

$$\gamma_o^k(G) \leq \frac{(r-1)n(G) + |L_k(G)|}{r}.$$

**Proof.** Clearly, the set  $L_k(G)$  is contained in every  $\gamma_o^k(G)$ -set. In the case that  $|L_k(G)| = |V(G)|$ , we are finished. In the remaining case that  $|L_k(G)| < |V(G)|$ , let  $V_1, V_2, \dots, V_r$  be a partition of the  $r$ -partite graph  $G - L_k(G)$  such that  $|V_1| \geq |V_2| \geq \dots \geq |V_r|$ , where  $V_i = \emptyset$  is possible for  $i \geq 2$ . Then every vertex of  $V_1$  has degree at least  $k$  in  $G$ , and all its neighbors are in  $V(G) - V_1$ . Thus  $V(G) - V_1$  is a global offensive  $k$ -alliance of  $G$ . Since

$$|V_1| \geq \frac{|V_1| + |V_2| + \dots + |V_r|}{r} = \frac{n(G) - |L_k(G)|}{r},$$

we obtain

$$\gamma_o^k(G) \leq n(G) - |V_1| \leq n(G) - \frac{n(G) - |L_k(G)|}{r} = \frac{(r-1)n(G) + |L_k(G)|}{r},$$

and the proof is complete. ■

The case  $k = r = 2$  in Theorem 1 leads to the next result.

**Corollary 2** (Chellali [4]). *If  $G$  is a bipartite graph, then*

$$\gamma_o^2(G) \leq \frac{n(G) + |L_2(G)|}{2}.$$

**Observation 3.** *If  $k \geq 1$  is an integer, then  $\gamma_o^k(G) \geq \gamma_k(G)$  for any graph  $G$ .*

**Proof.** If  $S$  is any  $\gamma_o^k(G)$ -set, then every vertex of  $V(G) - S$  has at least  $k$  neighbors in  $S$ . Thus  $S$  is a  $k$ -dominating set of  $G$  and so  $\gamma_k(G) \leq |S| = \gamma_o^k(G)$ . ■

Using Theorem 1 for  $r = 2$  and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].

**Corollary 4** (Blidia, Chellali, Volkmann [2] 2006). *Let  $k$  be a positive integer. If  $G$  is a bipartite graph, then*

$$\gamma_k(G) \leq \frac{n(G) + |L_k(G)|}{2}.$$

Since every graph  $G$  is  $\chi(G)$ -partite and  $n(G) \leq \chi(G)\alpha(G)$ , we obtain also the following corollaries from Theorem 1.

**Corollary 5.** *If  $G$  is a graph and  $k$  a positive integer, then*

$$\gamma_o^k(G) \leq \frac{(\chi(G) - 1)n(G) + |L_k(G)|}{\chi(G)}.$$

**Corollary 6.** *Let  $k \geq 1$  be an integer. If  $G$  is a graph with  $\delta(G) \geq k$ , then*

$$\gamma_o^k(G) \leq (\chi(G) - 1)\alpha(G).$$

**Theorem 7** (Brooks [3] 1941). *If  $G$  is a connected graph different from the complete graph and from a cycle of odd length, then  $\chi(G) \leq \Delta(G)$ .*

Combining Brooks' Theorem and Corollary 6, we can prove the following result.

**Theorem 8.** *Let  $k \geq 1$  be an integer, and let  $G$  be a connected graph with  $\delta(G) \geq k$ . Then*

$$(1) \quad \gamma_o^k(G) \leq (\Delta(G) - 1)\alpha(G)$$

*if and only if  $G$  is neither isomorphic to the complete graphs  $K_{k+1}$  or  $K_{k+2}$  nor to a cycle of odd length when  $1 \leq k \leq 2$ .*

**Proof.** If  $G$  is the complete graph  $K_n$ , then  $\Delta(G) = \delta(G) = n - 1 \geq k \geq 1$  and  $\alpha(G) = 1$ . Since  $\gamma_o^k(K_{k+1}) = k$  and  $\gamma_o^k(K_{k+2}) = k + 1$ , inequality (1) is not true for these two complete graphs. However, in the remaining case that  $n \geq k + 3$ , we observe that  $\gamma_o^k(G) \leq n - 2$ , and we arrive at the desired bound

$$\gamma_o^k(G) \leq n - 2 = \Delta(G) - 1 = (\Delta(G) - 1)\alpha(G).$$

Assume next that  $1 \leq k \leq 2$ . If  $G$  is a cycle of odd length, then  $\Delta(G) = 2$ ,  $\gamma_o^1(G) = \gamma_o^2(G) = \lceil n(G)/2 \rceil$  and  $\alpha(G) = \lfloor n(G)/2 \rfloor$  and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6. ■

**Lemma 9** (Hansberg, Meierling, Volkmann [10]). *Let  $k \geq 1$  be an integer. If  $G$  is a connected graph with  $\delta(G) \leq k - 1$  and  $\Delta(G) \leq k$ , then*

$$k\alpha(G) \geq n(G).$$

**Theorem 10.** *Let  $k \geq 1$  be an integer. If  $G$  is a connected  $r$ -partite graph with  $\Delta(G) \geq k$ , then*

$$\gamma_o^k(G) \leq \frac{\alpha(G)}{r}((r - 1)r + k - 1).$$

**Proof.** Assume that  $k = 1$ . Since  $G$  is connected and  $\Delta(G) \geq 1$ , we note that  $|L_1(G)| = 0$ . Applying Theorem 1, and using the fact that  $r\alpha(G) \geq n(G)$ , we receive the desired inequality immediately.

Assume next that  $k \geq 2$ . Since  $G$  is connected and  $G - L_k(G)$  is not empty, every component  $Q$  of  $G[L_k(G)]$  fulfills  $\delta(Q) \leq k - 2$  and  $\Delta(Q) \leq k - 1$ . Hence Lemma 9 implies  $(k - 1)\alpha(Q) \geq n(Q)$ . If  $Q_1, Q_2, \dots, Q_t$  are the components of  $G[L_k(G)]$ , we therefore deduce that

$$\alpha(G) \geq \alpha(G[L_k(G)]) = \sum_{i=1}^t \alpha(Q_i) \geq \frac{|L_k(G)|}{k - 1}.$$

Combining  $n(G) \leq r\alpha(G)$  with Theorem 1, we receive the desired inequality as follows:

$$\begin{aligned} \gamma_o^k(G) &\leq \frac{(r - 1)n(G) + |L_k(G)|}{r} \\ &\leq \frac{(r - 1)r\alpha(G) + (k - 1)\alpha(G)}{r} \\ &= \frac{\alpha(G)}{r}((r - 1)r + k - 1). \end{aligned}$$
■

The case  $r = 2$  in Theorem 10 leads to the next result.

**Corollary 11.** *Let  $k \geq 1$  be an integer. If  $G$  is a connected bipartite graph with  $\Delta(G) \geq k$ , then*

$$\gamma_o^k(G) \leq \frac{(k+1)\alpha(G)}{2}.$$

Using Observation 3, we obtain the following known bounds on the 2-domination number.

**Corollary 12** (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008). *If  $G$  is a connected bipartite graph of order at least 3, then*

$$\gamma_2(G) \leq \frac{3\alpha(G)}{2}.$$

**Corollary 13** (Blidia, Chellali, Favaron [1] 2005). *If  $T$  is a tree of order at least 3, then*

$$\gamma_2(T) \leq \frac{3\alpha(T)}{2}.$$

**Theorem 14** (Favaron, Hansberg, Volkmann [6] 2008). *Let  $G$  be a graph. If  $r \geq 1$  is an integer, then there is a partition  $V(G) = V_1 \cup V_2 \cup \dots \cup V_r$  of  $V(G)$  such that*

$$(2) \quad |N_G(u) \cap V_i| \leq \frac{d_G(u)}{r}$$

for each  $i \in \{1, 2, \dots, r\}$  and each  $u \in V_i$ .

**Theorem 15.** *Let  $k \geq 1$  be an integer. If  $G$  is a graph of order  $n$  with minimum degree  $\delta \geq k$ , then*

$$(3) \quad \gamma_o^k(G) \leq \frac{k+1}{k+2}n,$$

and the bound given in (3) is best possible.

**Proof.** Choose  $r = k + 2$  in Theorem 14, and let  $V_1, V_2, \dots, V_r$  be a partition of  $V(G)$  as in Theorem 14 such that  $|V_1| \geq |V_2| \geq \dots \geq |V_r|$ . If  $D = V_2 \cup V_3 \cup \dots \cup V_r$ , then it follows from (2) and the hypothesis that  $\delta \geq k$  for each  $v \in V_1 = V(G) - D$  that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k+1}{k+2}d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+2} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus  $D$  is a global offensive  $k$ -alliance of  $G$  such that  $|D| \leq (k+1)n/(k+2)$ , and (3) is proved.

Let  $H$  be a connected graph, and let  $G_k = H \circ K_{k+1}$ . Then it is easy to see that  $\gamma_o^k(G_k) = (k+1)n(G_k)/(k+2)$ , and therefore (3) is the best possible. ■

**Corollary 16** (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004). *Let  $G$  be graph of order  $n$  and minimum degree  $\delta$ .*

*If  $\delta \geq 1$ , then  $\gamma_o^1(G) \leq 2n/3$ .*

*If  $\delta \geq 2$ , then  $\gamma_o^2(G) \leq 3n/4$ .*

In the case that  $\delta \geq k+2$ , we obtain the following bound, improving the bound of Theorem 15.

**Theorem 17.** *Let  $k \geq 2$  be an integer, and let  $G$  be a graph of order  $n$  with minimum degree  $\delta \geq k+2$ . Then*

$$(4) \quad \gamma_o^k(G) \leq \frac{k}{k+1}n.$$

**Proof.** Choose  $r = k+1$  in Theorem 14, and let  $V_1, V_2, \dots, V_r$  be a partition of  $V(G)$  as in Theorem 14 such that  $|V_1| \geq |V_2| \geq \dots \geq |V_r|$ . If  $D = V_2 \cup V_3 \cup \dots \cup V_r$ , then it follows from (2) and the hypothesis  $\delta \geq k+2$  for each  $v \in V_1 = V(G) - D$  that

$$\begin{aligned} |N_G(v) \cap D| &\geq \left\lceil \frac{k}{k+1}d_G(v) \right\rceil \geq \left\lfloor \frac{d_G(v)}{k+1} \right\rfloor + k \\ &\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k. \end{aligned}$$

Thus  $D$  is a global offensive  $k$ -alliance of  $G$  such that  $|D| \leq kn/(k+1)$ , and (4) is proved. ■

**Theorem 18.** *Let  $k \geq 1$  be an integer, and let  $G$  be a connected non-complete graph such that  $\delta(G) \geq k$  and  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ . Then  $\Delta(G) \leq k+2$ ,  $\Delta(G) - \delta(G) \leq 1$  and if  $k \geq 2$ , then  $\delta(G) \leq k+1$ .*

**Proof.** Because of  $\chi(G)\alpha(G) \geq n(G)$ , Corollary 5 and the hypothesis imply that

$$(\Delta(G) - 1)\alpha(G) = \gamma_o^k(G) \leq \frac{(\chi(G) - 1)n(G)}{\chi(G)} \leq (\chi(G) - 1)\alpha(G).$$

Since  $G$  is neither a complete graph nor a cycle of odd length, it follows from Brooks' Theorem that  $\Delta(G) = \chi(G)$ ,  $\chi(G)\alpha(G) = n(G)$  and

$$(5) \quad \gamma_o^k(G) = \frac{(\chi(G) - 1)n(G)}{\chi(G)} = \frac{(\Delta(G) - 1)n(G)}{\Delta(G)}.$$

If we suppose on the contrary that  $\Delta(G) \geq k + 3$ , then it follows from (5) and Theorem 15 that

$$\frac{\Delta(G) - 1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k + 1}{k + 2}n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1}n(G).$$

This contradiction shows that  $\Delta(G) \leq k + 2$ .

If we suppose on the contrary that  $\Delta(G) - \delta(G) \geq 2$ , then we deduce that  $\delta(G) = k$  and  $\Delta(G) = k + 2 = \chi(G)$ . Since  $\chi(G)\alpha(G) = n(G)$ , there exists a partition of  $V(G)$  in  $\chi = \chi(G)$  colour classes  $U_1, U_2, \dots, U_\chi$  such that  $|U_1| = |U_2| = \dots = |U_\chi| = \alpha(G)$ . Let  $v$  be a vertex of minimum degree  $\delta(G) = k$ , and assume, without loss of generality, that  $v \in U_1$ . As  $d_G(v) = k$  and  $\chi(G) = k + 2$ , there exists a colour class  $U_j$  with  $2 \leq j \leq \chi$  such that  $v$  is not adjacent to any vertex in  $U_j$ . Therefore  $U_j \cup \{v\}$  is an independent set. This is a contradiction to the fact that  $|U_j| = \alpha(G)$ , and the desired inequality  $\Delta(G) - \delta(G) \leq 1$  is proved.

Next assume that  $k \geq 2$ , and suppose on the contrary that  $\delta(G) \geq k + 2$ . Then  $k \leq \Delta(G) - 2$  and (5) and Theorem 17 lead to the contradiction

$$\frac{\Delta(G) - 1}{\Delta(G)}n(G) = \gamma_o^k(G) \leq \frac{k}{k + 1}n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1}n(G).$$

Thus  $\delta(G) \leq k \leq \delta(G) + 1$  when  $k \geq 2$ , and the proof of Theorem 18 is complete. ■

**Example 19.** 1. Let  $H_1, H_2, \dots, H_t$  be  $t \geq 2$  copies of the complete graph  $K_{k+1}$ , and let  $u_i, v_i \in E(H_i)$  for  $1 \leq i \leq t$ . Define the graph  $G$  as the disjoint union  $H_1 \cup H_2 \cup \dots \cup H_t$  together with the edge set  $\{v_1u_2, v_2u_3, \dots, v_{t-1}u_t\}$ . Then it is easy to verify that  $\Delta(G) = k + 1$ ,  $\delta(G) = k$ ,  $\alpha(G) = t$ ,  $\gamma_o^k(G) = tk$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

2. Let  $F_1$  and  $F_2$  be 2 copies of the complete graph  $K_{k+1}$  with the vertex sets  $V(F_1) = \{x_1, x_2, \dots, x_{k+1}\}$  and  $V(F_2) = \{y_1, y_2, \dots, y_{k+1}\}$ . Define the graph  $H$  as the disjoint union  $F_1 \cup F_2$  together with the edge set  $\{x_1y_1, x_2y_2, \dots, x_ky_k\}$ . If  $H_1, H_2, \dots, H_t$  are  $t \geq 2$  copies of  $H$ , then let



$u_{2i-1}$  and  $u_{2i}$  be the vertices of degree  $k$  in  $H_i$  for all  $i \in \{1, 2, \dots, t\}$ . Define the graph  $G$  as the disjoint union  $H_1 \cup H_2 \cup \dots \cup H_t$  together with the edge set  $\{u_2u_3, u_4u_5, \dots, u_{2t}u_1\}$ . Then  $G$  is a  $(k + 1)$ -regular graph with  $\alpha(G) = 2t$ ,  $\gamma_o^k(G) = 2kt$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

3. Let  $k \geq 2$ , and let  $F_1$  and  $F_2$  be 2 copies of the complete graph  $K_k$  such that  $V(F_1) = \{x_1, x_2, \dots, x_k\}$  and  $V(F_2) = \{y_1, y_2, \dots, y_k\}$ . Define the graph  $H$  as the disjoint union  $F_1 \cup F_2$  together with the edge set  $\{x_1y_1, x_2y_2, \dots, x_{k-1}y_{k-1}\}$ . If  $H_1, H_2, \dots, H_t$  are  $t \geq 2$  copies of  $H$ , then let  $u_{2i-1}$  and  $u_{2i}$  be the vertices of degree  $k - 1$  in  $H_i$  for all  $i \in \{1, 2, \dots, t\}$ . Define the graph  $G$  as the disjoint union  $H_1 \cup H_2 \cup \dots \cup H_t$  together with the edge set  $\{u_2u_3, u_4u_5, \dots, u_{2t}u_1\}$ . Then  $G$  is a  $k$ -regular graph with  $\alpha(G) = 2t$ ,  $\gamma_o^k(G) = 2(k - 1)t$  and thus  $\gamma_o^k(G) = (\Delta(G) - 1)\alpha(G)$ .

4. Let  $H_1$  and  $H_2$  be 2 copies of the complete graph  $K_{k+2}$ , and let  $x \in E(H_1)$  and  $y \in E(H_2)$ . Define the graph  $G'$  as the disjoint union  $H_1 \cup H_2$  together with the edge  $xy$ . Then  $\Delta(G') = k + 2$ ,  $\delta(G') = k + 1$ ,  $\alpha(G') = 2$ ,  $\gamma_o^k(G') = 2(k + 1)$  and thus  $\gamma_o^k(G') = (\Delta(G') - 1)\alpha(G')$ .

These four examples show that  $\Delta = k + 1$  and  $\delta = k$ ,  $\Delta = \delta = k + 1$ ,  $\Delta = \delta = k$  as well as  $\Delta = k + 2$  and  $\delta = k + 1$  in Theorem 18 are possible.

**Theorem 20.** *If  $G$  is a graph and  $k$  an integer such that  $1 \leq k \leq \delta(G) - 1$ , then*

$$\gamma_o^{k+1}(G) \leq \frac{\gamma_o^k(G) + n(G)}{2}.$$

**Proof.** Let  $S$  be a  $\gamma_o^k(G)$ -set, and let  $A$  be the set of isolated vertices in the subgraph induced by the vertex set  $V(G) - S$ . Then the subgraph induced by  $V(G) - (S \cup A)$  contains no isolated vertices. If  $D$  is a minimum dominating set of  $G[V(G) - (S \cup A)]$ , then the well-known inequality of Ore [12] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_o^k(G)}{2}.$$

Since  $\delta(G) \geq k + 1$ , every vertex of  $A$  has at least  $k + 1$  neighbors in  $S$ , and therefore  $D \cup S$  is a global offensive  $(k + 1)$ -alliance of  $G$ . Thus we obtain the desired bound as follows:

$$\gamma_o^{k+1}(G) \leq |S \cup D| \leq \gamma_o^k(G) + \frac{n(G) - \gamma_o^k(G)}{2} = \frac{\gamma_o^k(G) + n(G)}{2}. \quad \blacksquare$$

The graphs  $G$  of even order and without isolated vertices with  $\gamma(G) = n/2$  have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

**Theorem 21** (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985). *Let  $G$  be a graph of even order  $n$  without isolated vertices. Then  $\gamma(G) = n/2$  if and only if each component of  $G$  is either a cycle  $C_4$  or the corona of a connected graph.*

A graph is  $P_4$ -free if and only if it contains no induced subgraph isomorphic to the path  $P_4$  of order four. A graph is  $(K_4 - e)$ -free if and only if it contains no induced subgraph isomorphic to the graph  $K_4 - e$ , where  $e$  is an arbitrary edge of the complete graph  $K_4$ . The graph  $\overline{G}$  denotes the complement of the graph  $G$ . Next we give a characterization of some special graphs attaining equality in Theorem 20.

**Theorem 22.** *Let  $G$  be a connected  $P_4$ -free graph such that  $\overline{G}$  is  $(K_4 - e)$ -free. If  $k$  is an integer with  $1 \leq k \leq \delta(G) - 1$ , then  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$  if and only if*

1.  $G = K_{k+3}$  or
2.  $\overline{G} = H \cup 2K_{1,1}$  such that  $n(H) = k + 2$  and all components of  $H$  are isomorphic to  $K_{1,1}$ , to  $K_{3,3}$ , to  $K_{3,4}$  or to  $K_{4,4}$  or
3.  $G = (Q_1 \cup Q_2) + F$ , where  $Q_1, Q_2$  and  $F$  are three pairwise disjoint graphs such that  $1 \leq |V(F)| \leq k + 1$ ,  $\alpha(F) \leq 2$ , and  $Q_1$  and  $Q_2$  are cliques with  $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$  such that
  - $|V(F)| \leq 2$  or
  - $\alpha(F) = 1$  and  $|V(F)| = k + 1$  or
  - $\alpha(F) = 2$  and  $F = K_{k+1} - M$ , where  $M$  is a matching of  $F$  or
  - $\alpha(F) = 2$  and  $F = K_k - M$ , where  $M$  is a perfect matching of  $F$  or
  - $\alpha(F) = 2$  and  $|V(F)| = k + 1 - t$  for  $0 \leq t \leq k - 2$  with  $k \geq 3t + 3$  and all components of  $\overline{F}$  are isomorphic to  $K_{t+2, t+2}$ , to  $K_{t+2, t+3}$  or to  $K_{t+3, t+3}$ .

**Proof.** Assume that  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ . Following the same notation as used in the proof of Theorem 20, we obtain  $|D| = \frac{|V(G) - S|}{2}$ , and we observe that  $S \cup D$  is a  $\gamma_o^{k+1}(G)$ -set. It follows that  $G[V(G) - S]$  has no isolated vertices and so by Theorem 21, each component of  $G[V(G) - S]$  is either a cycle  $C_4$  or the corona of some connected graph. Using the

hypothesis that  $G$  is  $P_4$ -free, we deduce that each component of  $G[V(G) - S]$  is isomorphic to  $K_2$  or to  $C_4$ . Since  $\overline{G}$  is  $(K_4 - e)$ -free, there remain exactly the three cases that  $G[V(G) - S]$  is isomorphic to  $K_2$ , to  $C_4$  or to  $2K_2$ .

*Case 1.* First assume that  $G[V(G) - S] = K_2$ . Suppose that  $G$  has an independent set  $Q$  of size at least two. Then the hypothesis  $\delta(G) \geq k + 1$  implies that  $V(G) - Q$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - |Q| < |S \cup D| = n - 1$ , a contradiction. Therefore  $\alpha(G) = 1$  and thus  $G = K_{k+3}$ .

*Case 2.* Second assume that  $G[V(G) - S]$  is a cycle  $C_4 = x_0x_1x_2x_3x_0$ . In the following the indices of the vertices  $x_i$  are taken modulo 4. Recall that  $S \cup D$  is a  $\gamma_o^{k+1}(G)$ -set, and  $D$  contains two vertices of the cycle  $C_4$ . Clearly, since  $S$  is a  $\gamma_o^k(G)$ -set, every vertex of the cycle  $C_4$  has degree at least  $k + 4$ . Suppose that  $d_G(x_i) \geq k + 5$  for an  $i \in \{0, 1, 2, 3\}$ . Then  $\{x_{i+2}\} \cup S$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $|S| + 1 < |S \cup D| = |S| + 2$ , a contradiction. Thus  $d_G(x_i) = k + 4$  for every  $i \in \{0, 1, 2, 3\}$ . Now if  $Q$  is an  $\alpha(G)$ -set, then  $|Q| \leq 2$ , for otherwise the hypothesis  $\delta(G) \geq k + 1$  implies that  $V(G) - Q$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $|V(G) - Q| < |S \cup D| = n(G) - 2$ , a contradiction too. Since there are two non-adjacent vertices on the cycle  $C_4$  and  $G$  is  $P_4$ -free, it follows that every vertex of  $S$  has at least three neighbors on the cycle  $C_4$ .

*Subcase 2.1.* Assume that  $\alpha(G[S]) = 1$ . Then the subgraph induced by  $S$  is complete and  $|S| \geq k + 2$ . If  $|S| = k + 2$ , then we observe that every vertex of  $S$  has exactly four neighbours on the cycle  $C_4$ . Thus, in each case, we deduce that  $d_G(y) \geq k + 5$  for every  $y \in S$ . But then for any subset  $W$  of  $S$  of size three, the set  $V(G) - W$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size less than  $|S \cup D|$ , a contradiction.

*Subcase 2.2.* Assume that  $\alpha(G[S]) = 2$ . Suppose that there exists a vertex  $w \in S$  with at least  $k + 1$  neighbors in  $S$ . Then, since  $|N(w) \cap V(C_4)| \geq 3$ , say  $\{x_0, x_1, x_2\} \subseteq N(w) \cap V(C_4)$ , we observe that  $(S - \{w\}) \cup \{x_0, x_2\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $|S| + 1 < |S \cup D|$ , a contradiction. Thus every vertex of  $S$  has at most  $k$  neighbors in  $S$ .

Let  $S = X \cup Y$  such that every vertex of  $X$  has exactly three and every vertex of  $Y$  exactly 4 neighbors on  $C_4$ . We shall show that  $X = \emptyset$ . If  $X \neq \emptyset$ , then let  $S_{x_i} \subseteq X$  be the set of vertices such that each vertex of  $S_{x_i}$  is not adjacent to  $x_{i+2}$  for  $i \in \{0, 1, 2, 3\}$ . Because of  $\alpha(G) = 2$ , we observe that

the set  $S_{x_i} \cup \overline{\{x_i\}}$  induces a complete graph for each  $i \in \{0, 1, 2, 3\}$ . In addition, since  $G$  is  $P_4$ -free it is straightforward to verify that all vertices of  $X \cup C_4$  are adjacent to all vertices of  $Y$  and that  $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$  induces a complete graph for each  $i \in \{0, 1, 2, 3\}$ . Now assume, without loss of generality, that  $S_{x_0} \neq \emptyset$ , and let  $w \in S_{x_0}$ . On the one hand we have seen above that  $d_G(w) \leq k + 3$ . On the other hand, we observe that  $d_G(w) = d_G(x_0)$ . But since  $d_G(x_0) = k + 4$ , we have a contradiction.

Hence we have shown that  $X = \emptyset$ , and this leads to  $|S| = k + 2$ . If we define  $H = \overline{G[S]}$ , then  $\omega(H) = 2$ ,  $\delta(H) \geq 1$  and  $\Delta(H) \leq 4$ . Since  $H$  is also  $P_4$ -free,  $H$  does not contain an induced cycle of odd length. Using  $\omega(H) = 2$ , we deduce that  $H$  is a bipartite graph. Now let  $H_i$  be a component of  $H$ . If  $H_i$  is not a complete bipartite graph, then  $H_i$  contains a  $P_4$ , a contradiction. Thus the components of  $H$  consists of  $K_{1,1}$ ,  $K_{1,2}$ ,  $K_{1,3}$ ,  $K_{1,4}$ ,  $K_{2,2}$ ,  $K_{2,3}$ ,  $K_{2,4}$ ,  $K_{3,3}$ ,  $K_{3,4}$  or  $K_{4,4}$ .

If  $K_{1,2}$  is a component of  $H$ , then  $V(G) - V(K_{1,2})$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

If  $K_{1,3}$  is a component of  $H$  with a leaf  $u$ , then  $(V(G) - V(K_{1,3})) \cup \{u\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

If  $K_{1,4}$  is a component of  $H$  and  $u, v$  are two leaves of this star, then  $(V(G) - V(K_{1,3})) \cup \{u, v\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

If  $K_{2,2}$  is a component of  $H$ , then  $V(G) - V(K_{2,2})$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 4$ , a contradiction.

Next let  $K_{2,3}$  be a component of  $H$  with the bipartition  $\{v_1, v_2, v_3\}$  and  $\{u_1, u_2\}$ . Then  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

Finally, let  $K_{2,4}$  be a component of  $H$  with the bipartition  $\{v_1, v_2, v_3, v_4\}$  and  $\{u_1, u_2\}$ . Then  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

*Case 3.* Third assume that  $G[V(G) - S] = 2K_2$ . Let  $2K_2 = J_1 \cup J_2 = J$  such that  $V(J_1) = \{u_1, u_2\}$  and  $V(J_2) = \{u_3, u_4\}$ . If  $\alpha(G) \geq 3$ , then we obtain the contradiction  $\gamma_o^{k+1}(G) \leq n - 3$ . Thus  $\alpha(G) = 2$ . Since  $S$  is a  $\gamma_o^k(G)$ -set, every vertex of  $J$  has degree at least  $k + 2$ . Suppose that  $d_G(u_1) \geq k + 3$  and  $d_G(u_2) \geq k + 3$ . Then  $\{u_3\} \cup S$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $|S| + 1 < |S \cup D| = |S| + 2$ , a contradiction. Thus  $J_1$  contains at least one vertex of degree  $k + 2$ , and for reason of symmetry, also  $J_2$  contains a vertex of degree  $k + 2$ . Since  $\alpha(G) = 2$ , every vertex of

$S$  has at least two neighbors in  $J_1$  or in  $J_2$ . Now let  $x \in S$ . If  $x$  has two neighbors in  $J_i$  and one neighbor in  $J_{3-i}$  for  $i = 1, 2$ , then the hypothesis that  $G$  is  $P_4$ -free implies that  $x$  is adjacent to each vertex of  $J$ . Consequently,  $S$  can be partitioned in three subsets  $S_1, S_2$  and  $A$  such that all vertices of  $S_1$  are adjacent to all vertices of  $J_1$  and there is no edge between  $S_1$  and  $J_2$ , all vertices of  $S_2$  are adjacent to all vertices of  $J_2$  and there is no edge between  $S_2$  and  $J_1$ , all vertices of  $A$  are adjacent to all vertices of  $J$ . Since  $G$  is  $P_4$ -free, it follows that there is no edge between  $S_1$  and  $S_2$ , and that all vertices of  $S_i$  are adjacent to all vertices of  $A$  for  $i = 1, 2$ . Furthermore,  $\alpha(G) = 2$  shows that  $G[S_1]$  and  $G[S_2]$  are cliques. Altogether we see that  $d_G(u_i) = k + 2$  for each  $i \in \{1, 2, 3, 4\}$  and therefore  $|S_1| + |A| = |S_2| + |A| = k + 1$ . It follows that  $|S_1| = |S_2|$  and  $|S| + |A| = 2k + 2$ . Since  $G$  is connected, we deduce that  $|A| \geq 1$  and so  $1 \leq |A| \leq k + 1$ . If we define  $F = G[A]$  and  $Q_i = G[S_i \cup V(J_i)]$  for  $i = 1, 2$ , then we derive the desired structure, since  $\alpha(G[A]) \leq 2$ .

Assume that  $|V(F)| \geq 3$  and  $\alpha(F) = 1$ . If  $x_1, x_2, x_3$  are three arbitrary vertices in  $F$ , then let  $S_0 = V(G) - \{x_1, x_2, x_3\}$ . If  $d_G(x_i) \geq k + 5$  for each  $i = 1, 2, 3$ , then  $S_0$  is a global offensive  $(k + 1)$ -alliance of  $G$ , a contradiction. Otherwise, we have  $n - 1 = d_G(x_i) \leq k + 4$  for at least one  $i \in \{1, 2, 3\}$  and so  $n \leq k + 5$  and thus  $|V(F)| = k + 1$ .

Assume next that  $|V(F)| \geq 3$  and  $\alpha(F) = 2$ . As we have seen in Case 2, all components of  $\overline{F}$  are complete bipartite graphs.

*Subcase 3.1.* Assume that  $K_{1,1}$  is the greatest component of  $\overline{F}$ . Let  $u$  and  $v$  be the two vertices of the complete bipartite graph  $K_{1,1}$ . If  $n \geq k + 7$ , then let  $w$  be a further vertex in  $F$ , and it is easy to verify that  $V(G) - \{u, v, w\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction. If  $n = k + 6$  and there exists a vertex  $w$  in  $F$  of degree  $k + 5$ , then  $V(G) - \{u, v, w\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

*Subcase 3.2.* Assume that  $|V(F)| = k + 1 - t$  for  $0 \leq t \leq k - 2$  and  $\overline{F}$  contains a component  $K_{p,q}$  with  $1 \leq p \leq q$  and  $p + q \geq 3$ . Let  $\{v_1, v_2, \dots, v_q\}$  and  $\{u_1, u_2, \dots, u_p\}$  be a partition of  $K_{p,q}$ .

If  $K_{1,s} \subseteq \overline{F}$  with  $s \geq t + 4$ , then  $\delta(G) \leq k$ , a contradiction to  $\delta(G) \geq k + 1$ . Thus  $q \leq t + 3$ .

If  $q \leq t + 1$  or  $q = t + 2$  and  $p \leq t + 1$ , then it is easy to see that  $V(G) - \{u_1, v_1, v_2\}$  is a global offensive  $(k + 1)$ -alliance of  $G$  of size  $n - 3$ , a contradiction.

Conversely, if  $G = K_{k+3}$ , then obviously  $\gamma_o^k(G) = k + 1$ ,  $\gamma_o^{k+1}(G) = k + 2$  and so  $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ .

Now let  $\overline{G} = H \cup 2K_{1,1}$  such that  $n(H) = k + 2$  and the components of  $H$  are complete bipartite graphs  $K_{1,1}$ ,  $K_{3,3}$ ,  $K_{3,4}$  or  $K_{4,4}$ . Thus  $k + 1 \leq d_G(z) \leq k + 4$  for every  $z \in V(G)$ , and  $G$  contains a cycle  $C$  on four vertices, where each vertex of  $C$  has degree  $k + 4$ . Clearly,  $V(H)$  is a global offensive  $k$ -alliance of  $G$  and so  $\gamma_o^k(G) \leq n(G) - 4$ . If  $D$  is a  $\gamma_o^k(G)$ -set of size  $|D| \leq n(G) - 5$ , then, since  $\alpha(G) = 2$ , the induced subgraph  $G[V(G) - D]$  contains a vertex  $x$  of degree at least two. This leads to the contradiction  $|N_G(x) \cap D| \leq k + 1 < |N_G(x) - D| + k$ . Hence we have shown that  $\gamma_o^k(G) = n(G) - 4$ .

Now let us prove that  $\gamma_o^{k+1}(G) = n(G) - 2$ . Clearly,  $\gamma_o^{k+1}(G) \geq \gamma_o^k(G) \geq n(G) - 4$ . Let  $D$  be a  $\gamma_o^{k+1}(G)$ -set. First, assume that  $\gamma_o^{k+1}(G) = n(G) - 4$ . Then, since  $n(G) = k + 6$  and  $\alpha(G) = 2$ , the induced subgraph  $G[V(G) - D]$  is isomorphic to  $2K_{1,1}$ , say  $ab$  and  $cd$ , and every vertex of  $V(G) - D$  is adjacent to all vertices of  $D$ . Since  $d_G(x) = k + 3$  for every  $x \in \{a, b, c, d\}$  it follows that  $a, b, c, d$  lie in one component  $C_4$  of  $H$ , a contradiction. Second, assume that  $\gamma_o^{k+1}(G) = n(G) - 3$ . Since every vertex has degree at most  $k + 4$ , no vertex of  $V(G) - D$  has two neighbors in  $V(G) - D$ . Moreover, since  $\alpha(G) = 2$ ,  $G[V(G) - D]$  is formed by two adjacent vertices  $x, y$  plus an isolated vertex  $w$ . Since  $w$  has degree at least two in  $\overline{G}$ , the vertices  $w, x, y$  lie in one component in  $H$  and so belong to  $K_{3,3}$ ,  $K_{3,4}$  or  $K_{4,4}$ . Thus each of  $x$  and  $y$  has at least two non-neighbors in  $D$  and hence  $|N(x) \cap D| \leq k + 1$ , a contradiction to the fact  $D$  is a  $\gamma_o^{k+1}(G)$ -set. Thus  $|D| \geq n(G) - 2$  and the equality follows from the fact that  $V(G)$  minus any two non-adjacent vertices of  $C$  is a global offensive  $(k + 1)$ -alliance of  $G$ . Therefore  $\gamma_o^{k+1}(G) = n(G) - 2 = (\gamma_o^k(G) + n(G))/2$ .

Finally, let  $G = (Q_1 \cup Q_2) + F$ , where  $Q_1, Q_2$  and  $F$  are three pairwise disjoint graphs such that  $1 \leq |V(F)| \leq k + 1$ ,  $\alpha(F) \leq 2$ , and  $Q_1$  and  $Q_2$  are cliques with  $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$  such that  $|V(F)| \leq 2$  or  $\alpha(F) = 1$  and  $|V(F)| = k + 1$  or

$\alpha(F) = 2$  and  $F = K_{k+1} - M$ , where  $M$  is matching of  $F$  or

$\alpha(F) = 2$  and  $F = K_k - M$ , where  $M$  is a perfect matching of  $F$  or

$\alpha(F) = 2$  and  $|V(F)| = k + 1 - t$  for  $0 \leq t \leq k - 2$  with  $k \geq 3t + 3$  and all components of  $\overline{F}$  are isomorphic to  $K_{t+2, t+2}$ , to  $K_{t+2, t+3}$  or to  $K_{t+3, t+3}$ .

Let  $D$  be a global offensive  $(k + 1)$ -alliance of  $G$ . Since each vertex of  $Q_i$  has degree  $k + 2$ , the set  $V(G) - D$  contains at most one vertex of  $Q_i$  for every  $i = 1, 2$ . Moreover, if  $(V(G) - D) \cap V(Q_i) \neq \emptyset$ , then  $V(F) \subseteq D$ .

Now suppose that  $\gamma_o^{k+1}(G) \leq n - 3$ , and assume, without loss of generality, that  $V(G) - D = \{u, v, w\}$ . Then as noted above  $V(Q_1) \cup V(Q_2) \subseteq D$ , and hence the vertices  $u, v, w$  belong to  $V(F)$ . It follows that  $|V(F)| \geq 3$ .

Obviously, we obtain a contradiction when  $\alpha(F) = 1$  and  $|V(F)| = k + 1$ .

Assume next that  $\alpha(F) = 2$ . This implies that at least two vertices of  $V(G) - D$  are adjacent in  $G$ .

First assume that  $F = K_k - M$ , where  $M$  is a perfect matching of  $F$ . Note that every vertex of  $V(F)$  has degree  $k + 4$ . Since  $M$  is perfect,  $\{u, v, w\}$  induces either a path  $P_3$  or a clique  $K_3$  with center vertex, say  $v$ , in  $G$ . But then  $v$  has a non-neighbor in  $D$  for which it is matched in  $M$ , and so  $v$  has exactly  $k + 2$  neighbors in  $D$  against two in  $V(G) - D$ , a contradiction.

Second assume that  $F = K_{k+1} - M$ , where  $M$  is a matching of  $F$ . Note that  $n = k + 5$  and  $|D| = k + 2$ . As above,  $\{u, v, w\}$  induces either a path  $P_3$  or a clique  $K_3$  with center vertex, say  $v$ , in  $G$ . But then  $v$  has at most  $k + 2$  neighbors in  $D$  against two in  $V(G) - D$ , a contradiction.

Assume now that  $\alpha(F) = 2$  and  $|V(F)| = k + 1 - t$  for  $0 \leq t \leq k - 2$  with  $k \geq 3t + 3$  and all components of  $\bar{F}$  are isomorphic to  $K_{t+2, t+2}$ , to  $K_{t+2, t+3}$  or to  $K_{t+3, t+3}$ . Note that in this case  $n = k + 5 + t$  and so  $|D| = n - 3 = k + 2 + t$ . Assume, without loss of generality, that  $u$  and  $v$  are adjacent in  $G$ . This leads to  $|N_G(u) \cap D| \leq (k + 5 + t) - (t + 2 + 2) = k + 1$ , a contradiction to the assumption that  $D$  is a global offensive  $(k + 1)$ -alliance of  $G$ .

Altogether, we have shown that  $\gamma_o^{k+1}(G) = n - 2$ . Finally, it is a simple matter to obtain  $\gamma_o^k(G) = n - 4$ , and the proof of Theorem 22 is complete. ■

### 3. LOWER BOUNDS

Our aim in this section is to give lower bounds on the global offensive  $k$ -alliance number of a graph in terms of its order  $n$ , minimum degree  $\delta$  and maximum degree  $\Delta$ .

**Theorem 23.** *Let  $k$  be a positive integer. If  $G$  is a graph of order  $n$ , minimum degree  $\delta$  and maximum degree  $\Delta$ , then*

$$(6) \quad \gamma_o^k(G) \geq \frac{n(\delta + k)}{2\Delta + \delta + k}.$$

**Proof.** If  $S$  is any  $\gamma_o^k(G)$ -set, then

$$\Delta \gamma_o^k(G) = \Delta |S| \geq \sum_{v \in S} d_G(v) \geq \sum_{v \in V(G) - S} \frac{d_G(v) + k}{2}$$

$$\geq |V(G) - S| \frac{\delta + k}{2} = (n - \gamma_o^k(G)) \frac{\delta + k}{2}.$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k) \geq n(\delta + k),$$

and (6) is proved. ■

**Theorem 24.** *Let  $k \geq 1$  be an integer, and let  $G$  be a graph of order  $n$ , minimum degree  $\delta$  and maximum degree  $\Delta$ . If  $\delta$  is even and  $k$  odd or  $\delta$  odd and  $k$  even, then*

$$(7) \quad \gamma_o^k(G) \geq \frac{n(\delta + k + 1)}{2\Delta + \delta + k + 1}.$$

**Proof.** If  $S$  is any  $\gamma_o^k(G)$ -set, then

$$\begin{aligned} \Delta \gamma_o^k(G) &= \Delta |S| \geq \sum_{v \in S} d_G(v) \\ &\geq \sum_{v \in V(G) - S, d_G(v) = \delta} \frac{d_G(v) + k + 1}{2} + \sum_{v \in V(G) - S, d_G(v) > \delta} \frac{d_G(v) + k}{2} \\ &\geq |V(G) - S| \frac{\delta + k + 1}{2} = (n - \gamma_o^k(G)) \frac{\delta + k + 1}{2}. \end{aligned}$$

This leads to

$$\gamma_o^k(G)(2\Delta + \delta + k + 1) \geq n(\delta + k + 1),$$

and (7) is proved. ■

**Example 25.** Let  $G$  be a  $k$ -regular bipartite graph of order  $n$  with the partite sets  $X$  and  $Y$ . Then

$$\gamma_0^k(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + k)}{2\Delta + \delta + k}$$

and

$$\gamma_0^{k-1}(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + (k - 1) + 1)}{2\Delta + \delta + (k - 1) + 1}$$

for  $k \geq 2$ . This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.



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