ON $\gamma$-LABELINGS OF TREES

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Abstract

Let $G$ be a graph of order $n$ and size $m$. A $\gamma$-labeling of $G$ is a one-to-one function $f : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ that induces a labeling $f' : E(G) \rightarrow \{1, 2, \ldots, m\}$ of the edges of $G$ defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of $G$. The value of a $\gamma$-labeling $f$ is $\text{val}(f) = \sum_{e \in E(G)} f'(e)$. The maximum value of a $\gamma$-labeling of $G$ is defined as

$\text{val}_{\max}(G) = \max\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}$;

while the minimum value of a $\gamma$-labeling of $G$ is

$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}$. 
The values $\text{val}_{\text{max}}(S_{p,q})$ and $\text{val}_{\text{min}}(S_{p,q})$ are determined for double stars $S_{p,q}$. We present characterizations of connected graphs $G$ of order $n$ for which $\text{val}_{\text{min}}(G) = n$ or $\text{val}_{\text{min}}(G) = n + 1$.

**Keywords:** $\gamma$-labeling, value of a $\gamma$-labeling.

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1. Introduction

For a graph $G$ of order $n$ and size $m$, a $\gamma$-labeling of $G$ is a one-to-one function $f: V(G) \to \{0, 1, 2, \ldots, m\}$ that induces a labeling $f': E(G) \to \{1, 2, \ldots, m\}$ of the edges of $G$ defined by

$$f'(e) = |f(u) - f(v)|$$

for each edge $e = uv$ of $G$.

Therefore, a graph $G$ of order $n$ and size $m$ has a $\gamma$-labeling if and only if $m \geq n - 1$. In particular, every connected graph has a $\gamma$-labeling. If the induced edge-labeling $f'$ of a $\gamma$-labeling $f$ is also one-to-one, then $f$ is a graceful labeling, one of the most studied of graph labelings. An extensive survey of graph labelings as well as their applications has been given by Gallian [2].

Each $\gamma$-labeling $f$ of a graph $G$ of order $n$ and size $m$ is assigned a value denoted by $\text{val}(f)$ and defined by

$$\text{val}(f) = \sum_{e \in E(G)} f'(e).$$

Since $f$ is a one-to-one function from $V(G)$ to $\{0, 1, 2, \ldots, m\}$, it follows that $f'(e) \geq 1$ for each edge $e$ in $G$ and so

$$\text{val}(f) \geq m. \quad (1)$$

Figure 1 shows nine $\gamma$-labelings $f_1, f_2, \ldots, f_9$ of the path $P_5$ of order 5 (where the vertex labels are shown above each vertex and the induced edge labels are shown below each edge). The value of each $\gamma$-labeling is shown in Figure 1 as well.

For a graph $G$ of order $n$ and size $m$, the maximum value of a $\gamma$-labeling of a graph $G$ is defined as

$$\text{val}_{\text{max}}(G) = \max \{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\};$$
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while the minimum value of a $\gamma$-labeling of $G$ is

$$\text{val}_{\min}(G) = \min\{\text{val}(f) : f \text{ is a } \gamma\text{-labeling of } G\}.$$ 

A $\gamma$-labeling $g$ of $G$ is a $\gamma$-max labeling if

$$\text{val}(g) = \text{val}_{\max}(G)$$

and a $\gamma$-labeling $h$ is a $\gamma$-min labeling if

$$\text{val}(h) = \text{val}_{\min}(G).$$

Since $\text{val}(f_1) = 4$ for the $\gamma$-labeling $f_1$ of $P_5$ shown in Figure 1 and the size of $P_5$ is 4, it follows that $f_1$ is a $\gamma$-min labeling of $P_5$. Although less clear, the $\gamma$-labeling $f_0$ shown in Figure 1 is a $\gamma$-max labeling. The concepts of a $\gamma$-labeling of a graph and the value of a $\gamma$-labeling were introduced in [1].

For a $\gamma$-labeling $f$ of a graph $G$ of size $m$, the complementary labeling $\overline{f} : V(G) \rightarrow \{0, 1, 2, \ldots, m\}$ of $f$ is defined by

$$\overline{f}(v) = m - f(v) \text{ for } v \in V(G).$$

Not only is $\overline{f}$ a $\gamma$-labeling of $G$ as well but $\text{val}(\overline{f}) = \text{val}(f)$. This gives us the following observation that appeared in [1].
Observation 1.1. Let \( f \) be a \( \gamma \)-labeling of a graph \( G \). Then \( f \) is a \( \gamma \)-max labeling (\( \gamma \)-min labeling) of \( G \) if and only if \( \overline{f} \) is a \( \gamma \)-max labeling (\( \gamma \)-min labeling).

A more general vertex labeling of a graph was introduced by Hegde in [3]. A vertex function \( f \) of a graph \( G \) is defined from \( V(G) \) to the set of nonnegative integers that induces an edge function \( f' \) defined by \( f'(e) = |f(u) - f(v)| \) for each edge \( e = uv \) of \( G \). Such a function is called a geodetic function of \( G \). A one-to-one geodetic function is a geodetic labeling of \( G \) if the induced edge function \( f' \) is also one-to-one. The following result was established by Hegde which provides an upper bound for \( \text{val_{max}}(G) \) (see [3]).

**Theorem (Hegde).** For any geodetic \( \gamma \)-labeling \( f \) of a graph \( G \) of order \( n \),

\[
\sum_{e \in E(G)} f'(e) \leq \sum_{i=0}^{n-1} (2i - n + 1)f(v_i).
\]

The following results were obtained in [1] for the paths \( P_n \) and stars \( K_{1,n-1} \) of order \( n \).

**Theorem A.** For each integer \( n \geq 2 \),

\[
\text{val_{min}}(P_n) = n - 1 \text{ and } \text{val_{max}}(P_n) = \left\lfloor \frac{n^2 - 2}{2} \right\rfloor.
\]

**Theorem B.** Let \( G \) be a connected graph of order \( n \) and size \( m \). Then

\[
\text{val_{min}}(G) = m \text{ if and only if } G \cong P_n.
\]

**Theorem C.** For each integer \( n \geq 3 \),

\[
\text{val_{min}}(K_{1,n-1}) = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor \text{ and } \text{val_{max}}(K_{1,n-1}) = \binom{n}{2}.
\]

**Theorem D.** For each integer \( n \geq 3 \),

\[
\text{val_{min}}(C_n) = 2(n - 1)
\]
and

\[
\text{val}_{\text{max}}(C_n) = \begin{cases} 
\frac{n(n + 2)}{2} & \text{if } n \text{ is even,} \\
\frac{(n - 1)(n + 3)}{2} & \text{if } n \text{ is odd.}
\end{cases}
\]

In this paper, we investigate \(\gamma\)-labelings of trees, beginning with double stars.

2. \(\gamma\)-Labelings of Double Stars

We now turn to the double star \(S_{p,q}\) containing central vertices \(u\) and \(v\) such that \(\deg u = p\) and \(\deg v = q\) and determine \(\text{val}_{\text{min}}(S_{p,q})\) and then \(\text{val}_{\text{max}}(S_{p,q})\).

**Proposition 2.1.** For integers \(p, q \geq 2\),

\[
\text{val}_{\text{min}}(S_{p,q}) = \left(\left\lfloor \frac{p}{2} \right\rfloor + 1\right)^2 + \left(\left\lfloor \frac{q}{2} \right\rfloor + 1\right)^2 - \left(n_p \left\lfloor \frac{p + 2}{2} \right\rfloor + n_q \left\lfloor \frac{q + 2}{2} \right\rfloor + 1\right),
\]

where

\[
n_p = \begin{cases} 
1 & \text{if } p \text{ is even,} \\
0 & \text{if } p \text{ is odd}
\end{cases}
\quad \text{and} \quad
n_q = \begin{cases} 
1 & \text{if } q \text{ is even,} \\
0 & \text{if } q \text{ is odd.}
\end{cases}
\]

**Proof.** Let \(N(u) = \{v, u_1, u_2, \ldots, u_{p-1}\}\) and \(N(v) = \{u, v_1, v_2, \ldots, v_{q-1}\}\). Since the proof is similar whether \(p\) and \(q\) are odd or even, we provide the proof in one of these four cases only, namely when \(p\) and \(q\) are odd. Let \(p = 2s + 1\) and \(q = 2t + 1\) for positive integers \(s\) and \(t\). Define a \(\gamma\)-labeling \(f\) of \(S_{p,q}\) by

\[
f(x) = \begin{cases} 
\frac{s}{2} & \text{if } x = u, \\
2s + t + 1 & \text{if } x = v, \\
i - 1 & \text{if } x = u_i, 1 \leq i \leq s, \\
i & \text{if } x = u_i, s + 1 \leq i \leq 2s, \\
2s + i & \text{if } x = v_i, 1 \leq i \leq t, \\
2s + i + 1 & \text{if } x = v_i, t + 1 \leq i \leq 2t.
\end{cases}
\]
Thus exactly two edges in \( \{ uu_i : 1 \leq i \leq 2s \} \) are labeled \( a \) for each integer \( a \) with \( 1 \leq a \leq s \) and exactly two edges in \( \{ vv_i : 1 \leq i \leq 2t \} \) are labeled \( b \) for each integer \( b \) with \( 1 \leq b \leq t \). Furthermore, the edge \( uv \) is labeled \( s + t + 1 \). Therefore,

\[
\text{val}(f) = (s + t + 1) + 2(1 + 2 + \ldots + s) + 2(1 + 2 + \ldots + t) \\
= (s + t + 1) + 2 \left( \frac{s + 1}{2} \right) + 2 \left( \frac{t + 1}{2} \right) = (s + 1)^2 + (t + 1)^2 - 1.
\]

Therefore,

\[
\text{val}_{\text{min}}(S_{p,q}) \leq (s + 1)^2 + (t + 1)^2 - 1.
\]

Next, consider an arbitrary \( \gamma \)-labeling \( g \) of \( S_{p,q} \). We may assume that \( g(u) < g(v) \); otherwise, we could consider the complementary \( \gamma \)-labeling \( \overline{g} \) of \( g \). We show that

\[
\text{val}(g) \geq (s + 1)^2 + (t + 1)^2 - 1.
\]

First, we make the following observations:

1. At most two edges in \( \{ uu_i : 1 \leq i \leq 2s \} \) can be labeled \( a \) for each integer \( a \) with \( 1 \leq a \leq s \) and this can occur only if the labels in \( \{ g(u) + a : 1 \leq i \leq s \} \) are available for the vertices \( u_i \) (\( 1 \leq a \leq 2s \)).

2. At most two edges in \( \{ vv_i : 1 \leq i \leq 2t \} \) can be labeled \( b \) for each integer \( b \) with \( 1 \leq b \leq t \) and this can occur only if the labels in \( \{ g(v) + b : 1 \leq i \leq 2t \} \) are available for the vertices \( v_i \) (\( 1 \leq i \leq 2t \)).

Therefore,

\[
\sum_{e \in E(G) - \{uv\}} g'(e) \geq 2 \left( \frac{s + 1}{2} \right) + 2 \left( \frac{t + 1}{2} \right).
\]

Thus if \( g'(uv) = g(v) - g(u) \geq s + t + 1 \), then

\[
\text{val}(g) \geq (s + t + 1) + 2 \left( \frac{s + 1}{2} \right) + 2 \left( \frac{t + 1}{2} \right) = (s + 1)^2 + (t + 1)^2 - 1.
\]
Suppose then that $g'(uv) = s + t + 1 - k$ for some integer $k$ with $1 \leq k \leq s + t$. Then there are $s + t - k$ vertices of $S_{p,q}$ that are labeled with integers between $g(u)$ and $g(v)$. Consequently, $s + t + k$ vertices of $S_{p,q}$ are assigned a label less than $g(u)$ or greater than $g(v)$, which implies that at least $k$ vertices of $S_{p,q}$ are assigned a label less than $g(u) - s$ or greater than $g(v) + t$. For each vertex $u_i$, $1 \leq i \leq 2s$, assigned a label less than $g(u) - s$,
\[
\sum_{i=1}^{2s} g'(uu_i) \text{ must exceed } 2\binom{s + 1}{2}
\]
by at least 1; while for each vertex $v_i$, $1 \leq i \leq 2s$, assigned a label greater than $g(v) + t$,
\[
\sum_{i=1}^{2t} g'(vv_i) \text{ must exceed } 2\binom{t + 1}{2}
\]
by at least 1. Therefore,
\[
\sum_{e \in E(G) - \{uv\}} g'(e) \geq 2\binom{s + 1}{2} + 2\binom{t + 1}{2} + k.
\]

However then,
\[
\text{val}(g) = g'(uv) + \sum_{e \in E(G) - \{uv\}} g'(e)
\geq (s + t + 1 - k) + \left[2\binom{s + 1}{2} + 2\binom{t + 1}{2} + k\right]
= (s + 1)^2 + (t + 1)^2 - 1.
\]

In general, $\text{val}(g) \geq (s + 1)^2 + (t + 1)^2 - 1$. Therefore, $\text{val}_{\text{min}}(S_{p,q}) = (s + 1)^2 + (t + 1)^2 - 1$.

**Theorem 2.2** For every pair $p, q$ of positive integers,
\[
\text{val}_{\text{max}}(S_{p,q}) = \frac{1}{2} \left[p^2 + q^2 + 4pq - 3p - 3q + 2\right].
\]
Proof. Let $u$ and $v$ be the central vertices of $S_{p,q}$, where $\deg u = p$ and $\deg v = q$, and let $f$ be the $\gamma$-labeling of $S_{p,q}$ in which we assign the label 0 to $u$, the label $p + q - 1$ to $v$, the labels 1, 2, \ldots, $q - 1$ to the end-vertices adjacent to $v$, and the labels $q, q+1, \ldots, p+q-2$ to the end-vertices adjacent to $u$. The value of $f$ is $(p^2 + q^2 + 4pq - 3p - 3q + 2)/2$, which is therefore a lower bound for $\text{val}_{\max}(S_{p,q})$.

We now show that $\text{val}_{\max}(S_{p,q}) \leq (p^2 + q^2 + 4pq - 3p - 3q + 2)/2$. First, we claim that $S_{p,q}$ has a $\gamma$-max labeling for which $\{f(u), f(v)\} = \{0, p + q - 1\}$. We verify this claim by induction on $p + q$. The claim is clearly true for $p + q = 2$. Assume that the claim is true for $p + q = k - 1$, where $k \geq 3$. Let $T = S_{p,q}$, where $p + q = k$. Let $f$ be a $\gamma$-max labeling of $T$. If $\{f(u), f(v)\} = \{0, p + q - 1\}$, then the claim is true. Suppose that at least one $f(u)$ and $f(v)$ is neither 0 nor $p + q - 1$. By Observation 1.1, we may assume that $f(w) = p + q - 1$ and $w \neq u, v$. The vertex $w$ is therefore an end-vertex of $T$. Let $x \in \{u, v\}$ be the vertex of $T$ that is adjacent to $w$. Then $T' = T - w$ is isomorphic to $S_{p',q'}$, where $p' + q' = k - 1$. By the inductive hypothesis, $T'$ has a $\gamma$-max labeling $g$ for which $\{g(u), g(v)\} = \{0, p + q - 2\}$. By Observation 1.1, we may assume that $g(x) = 0$. Now

\[ \text{val}(f) = (p + q - 1 - f(x)) + \sum_{e \in E(T')} f'(e) \leq p + q - 1 + \text{val}_{\max}(T'). \]

We extend $g$ to a $\gamma$-labeling $h$ of $T$ by defining $h(w) = p + q - 1$. Then

\[ \text{val}(h) = p + q - 1 + \text{val}_{\max}(T'). \]

By (2) and (3), $\text{val}(f) \leq \text{val}(h)$. Since $f$ is a $\gamma$-max labeling of $T$, so too is $h$ a $\gamma$-max labeling of $T$. Let $y \in \{u, v\}$ for which $h(y) = p + q - 2$. Thus $y$ is not adjacent to $w$. Next, let $\phi$ be the $\gamma$-labeling of $T$ defined by

\[ \phi(z) = \begin{cases} h(z) & \text{if } z \neq w, y, \\ p + q - 1 & \text{if } z = y, \\ p + q - 2 & \text{if } z = w. \end{cases} \]

Then $\text{val}(\phi) = \text{val}(h)$ if $\deg y \leq 2$; while $\text{val}(\phi) > \text{val}(h)$ if $\deg y \geq 3$. Since $\text{val}(\phi)$ cannot exceed $\text{val}(h)$, it follows that $\deg y \leq 2$, and $\phi$ has the desired property that verifies the claim. By the claim and Observation 1.1, there is a $\gamma$-max labeling $f$ of $S_{p,q}$ with $f(u) = 0$ and $f(v) = p + q - 1$. 

If there is an end-vertex $t_1$ of $S_{p,q}$ adjacent to $v$ with $f(t_1) = i > q - 1$, then there is an end-vertex $t_2$ of $S_{p,q}$ adjacent to $u$ with $f(t_2) = j$, where $1 \leq j \leq q - 1$. Interchanging the labels of $t_1$ and $t_2$ produces a $\gamma$-labeling $f_1$ with $\text{val}(f_1) > \text{val}(f)$, which is impossible. Thus $f$ is the $\gamma$-labeling described in the first paragraph of the proof and $\text{val}(f) = (p^2 + q^2 + 4pq - 3p - 3q + 2)/2$.

3. Connected Graphs of Order $n$ with Minimum Value $n$

We already mentioned (in Theorem B) that a connected graph $G$ of order $n$ has minimum value $n - 1$ if and only if $G \cong P_n$. We now determine all those connected graphs $G$ of order $n$ for which $\text{val}_{\text{min}}(G) = n$. It is useful to present several lemmas first.

**Lemma 3.1.** If $G$ is a connected graph of size $m$ and $G'$ is a connected subgraph of $G$ having size $m'$, then

$$\text{val}_{\text{min}}(G) \geq (m - m') + \text{val}_{\text{min}}(G').$$

**Proof.** Suppose that $G$ has order $n$ and $G'$ has order $n'$. Let $f$ be a $\gamma$-min labeling of $G$. Then the restriction $h$ of $f$ to $G'$ is a one-to-one function. Suppose that the vertices of $G'$ are labeled $a_1, a_2, \ldots, a_{n'}$ by $h$, where $0 \leq a_1 < a_2 < \cdots < a_{n'} \leq m$. Thus, for $1 \leq i \neq j \leq n'$, $|a_i - a_j| \geq |i - j|$. Consider the one-to-one function $g : \{a_1, a_2, \ldots, a_{n'}\} \rightarrow \{0, 1, 2, \ldots, m'\}$ defined by $g(a_i) = i - 1$ for $1 \leq i \leq n'$. Then $\phi = g \circ h : V(G') \rightarrow \{0, 1, 2, \ldots, m'\}$ is a $\gamma$-labeling of $G'$. Furthermore,

$$\text{val}_{\text{min}}(G') \leq \text{val}(\phi) \leq \sum_{e \in E(G')} h'(e) = \sum_{e \in E(G')} f'(e).$$

Since $f'(e) \geq 1$ for every edge $e$ in $G$, it follows that

$$\text{val}(f) = \sum_{e \in E(G - G')} f'(e) + \sum_{e \in E(G')} f'(e) \geq (m - m') + \text{val}_{\text{min}}(G'),$$

as desired.

Lemma 3.1 can be extended to obtain the following result.
Lemma 3.2. If $G$ is a connected graph of size $m$ containing pairwise edge-disjoint connected subgraphs $G_1, G_2, \ldots, G_k$, where $G_i$ has size $m_i$ for $1 \leq i \leq k$, then

$$\text{val}_{\text{min}}(G) \geq \left( m - \sum_{i=1}^{k} m_i \right) + \sum_{i=1}^{k} \text{val}_{\text{min}}(G_i).$$

Lemma 3.3. Let $G$ be a connected graph of order $n$ with maximum degree $\Delta$. Then

$$\text{val}_{\text{min}}(G) \geq \begin{cases} (n - 1) + k(k - 1) & \text{if } \Delta = 2k, \\ (n - 1) + k^2 & \text{if } \Delta = 2k + 1. \end{cases}$$

Furthermore, this bound is sharp for stars.

Proof. Let $v \in V(G)$ with $\text{deg} v = \Delta$ and let $f$ be a $\gamma$-min labeling of $G$. Note that at most two edges incident with $v$ can be labeled $i$ for each $i$ with $1 \leq i \leq \lfloor \Delta/2 \rfloor$. Thus, if $\Delta = 2k$, then

$$\text{val}_{\text{min}}(G) \geq (n - 1 - 2k) + 2(1 + 2 + \cdots + k) = (n - 1) + k(k - 1);$$

while if $\Delta = 2k + 1$, then

$$\text{val}_{\text{min}}(G) \geq [(n - 1) - (2k + 1)] + 2(1 + 2 + \cdots + k) + (k + 1) = (n - 1) + k^2.$$

That this bound is sharp for stars follows from Theorem C. □

The proof of the next lemma is straightforward and is therefore omitted.

Lemma 3.4. Let $f$ be a $\gamma$-labeling of a connected graph $G$. If $P$ is a $u - v$ path in $G$, then

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)|.$$

Lemma 3.5. For the tree $F$ of Figure 2, $\text{val}_{\text{min}}(F) = 8$.

Proof. The $\gamma$-labeling $f$ of $F$ shown in Figure 2 has value 8 and so $\text{val}_{\text{min}}(F) \leq 8$. On the other hand, let $g$ be $\gamma$-min labeling of $F$ and
let \( u, v \in V(F) \) such that \( g(u) = 0 \) and \( g(v) = 6 \). Suppose that \( P \) is a \( u-v \) path in \( F \). Then

\[
\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = 6
\]

by Lemma 3.4. Since there are at least two edges of \( F \) not in \( P \), it follows that \( \text{val}_{\text{min}}(F) = \text{val}(g) \geq 8 \).

A caterpillar is a tree the removal of whose vertices results in a path. We are now able to characterize all connected graphs of order \( n \geq 4 \) whose minimum value is \( n \).

**Theorem 3.6.** Let \( G \) be a connected graph of order \( n \geq 4 \). Then \( \text{val}_{\text{min}}(G) = n \) if and only if \( G \) is a caterpillar, \( \Delta(G) = 3 \), and \( G \) has a unique vertex of degree 3.

**Proof.** Let \( T \) be the tree obtained from the path \( v_1, v_2, \ldots, v_{n-1} \) by adding the vertex \( v_n \) and joining \( v_n \) to a vertex \( v_k \), where \( 2 \leq k \leq n-2 \). Thus \( v_k \) is the only vertex of degree 3 in \( T \). Define a \( \gamma \)-labeling \( f \) of \( T \) by

\[
f(v_i) = \begin{cases} i - 1 & \text{if } 1 \leq i \leq k, \\ i & \text{if } k < i \leq n-1, \\ k & \text{if } i = n. \end{cases}
\]

Since \( \text{val}(f) = n \), it follows that \( \text{val}_{\text{min}}(T) \leq n \) and so \( \text{val}_{\text{min}}(T) = n \) by Theorem B.

For the converse, let \( G \) be a connected graph of order \( n \geq 4 \) such that \( G \) is not a caterpillar with \( \Delta(G) = 3 \) containing a unique vertex of degree 3. We show that \( \text{val}_{\text{min}}(G) \neq n \). This is certainly true if \( G \cong P_n \) or if \( G \) is not a tree by Theorem B. Hence we may assume that \( G \) is a tree \( T \) with...
If $\Delta(T) \geq 4$, then $val_{\text{min}}(T) \geq (n-1)+2 = n+1$ by Lemma 3.3. Thus $\Delta(T) = 3$. We consider two cases.

**Case 1.** $T$ contains two vertices $u$ and $v$ with degree 3. If $u$ and $v$ are adjacent, then $T$ contains the double star $S_{3,3}$ as a subgraph. By Theorem 2.2, $val_{\text{min}}(S_{3,3}) = 7$. Since the order of $S_{3,3}$ is 6, it then follows by Lemma 3.1 that $val_{\text{min}}(T) \geq (n-6)+7 = n+1$.

Thus we may assume that $u$ and $v$ are not adjacent. Let $N(u) = \{u_1, u_2, u_3\}$ and $N(v) = \{v_1, v_2, v_3\}$. Then $v \notin N(u)$ and $u \notin N(v)$. For any $\gamma$ labeling $g$ of $T$, $g'(e) \geq 2$ for at least one edge $e$ in $\{uu_i : 1 \leq i \leq 3\}$ and at least one edge $c$ in $\{vv_i : 1 \leq i \leq 3\}$. Therefore, at least two edges in $T$ are labeled 2 or more by $g$ and so $val_{\text{min}}(T) \geq val(g) \geq n+1$.

**Case 2.** $T$ has exactly one vertex of degree 3. Thus $T$ contains the graph $F$ in Lemma 3.5 as a subgraph. Since $val_{\text{min}}(F) = 8$ by Lemma 3.5 and the order of $F$ is 7, it then follows by Lemma 3.1 that $val_{\text{min}}(T) \geq (n-7)+8 = n+1$.

4. Some Results on the Minimum Value of a Tree in Terms of Its Order and Other Parameters

In Theorem 3.6, we considered caterpillars $T$ having maximum degree 3 and a unique vertex of degree 3. We now compute the minimum value of all such trees that are not necessarily caterpillars.

**Theorem 4.1.** Let $T$ be a tree of order $n \geq 4$ such that $\Delta(T) = 3$ and $T$ has a unique vertex $v$ of degree 3. If $d$ is the distance between $v$ and a nearest end-vertex, then

$$val_{\text{min}}(T) = n + d - 1.$$  

**Proof.** Let $x$, $y$, and $z$ be the three end-vertices of $T$, where $d(v,x) = d$, $d(v,y) = d'$, and $d(v,z) = d''$, where $d \leq d' \leq d''$. Let $P : v = v_0, v_1, \ldots, v_d = x$, $P' : v = u_0, u_1, \ldots, u_{d'} = y$, and $P'' : v = v_0, w_1, \ldots, w_{d''} = z$ denote the $v-x$ path, $v-y$ path, and $v-z$ path in $T$. Let $f : V(T) \rightarrow \{0,1,2,\ldots,n-1\}$ be the $\gamma$-labeling of $T$ for which $f(w_i) = d'' - i$ for $0 \leq i \leq d''$, $f(u_i) = d'' + i$ for $1 \leq i \leq d$, and $f(w_i) = i - d' + n - 1$ for $1 \leq i \leq d'$. Since $val(f) = n + d - 1$, it follows that $val_{\text{min}}(T) \leq n + d - 1$. 


On $\gamma$-Labelings of Trees

It remains therefore to show that $\text{val}_{\min}(T) \geq n + d - 1$. Let $g : V(T) \rightarrow \{0, 1, 2, \cdots, n-1\}$ be an arbitrary $\gamma$-labeling of $T$, and suppose that $g(v) = i$. Let

$$S = \{u \in V(T) : d(u, v) \leq d\}.$$ 

Thus $|S| = 3d + 1$. Let $a$ denote the smallest label assigned by $g$ to a vertex of $S$ and let $b$ denote the largest such label. We now consider two cases.

**Case 1.** The vertices in $S$ labeled $a$ and $b$ belong to two of the three paths $P$, $P'$, and $P''$, say $P$ and $P'$, respectively. Then

$$\sum_{e \in E(P)} g'(e) \geq i - a \quad \text{and} \quad \sum_{e \in E(P')} g'(e) \geq b - i.$$ 

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (i - a) + (b - i) + d = b - a + d \geq 3d + d = 4d.$$ 

Since there are $(n - 1) - 3d$ edges of $T$ not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 4d + (n - 1 - 3d) = n + d - 1.$$ 

**Case 2.** The vertices in $S$ labeled $a$ and $b$ belong to one of the three paths $P$, $P'$, and $P''$, say $P$. Then

$$\sum_{e \in E(P)} g'(e) \geq b - a.$$ 

Thus

$$\sum_{e \in \langle S \rangle} g'(e) \geq (b - a) + 2d \geq 3d + 2d = 5d.$$ 

Since there are $(n - 1) - 3d$ edges of $T$ not belonging to $\langle S \rangle$, it follows that

$$\sum_{e \in E(T)} g'(e) \geq 5d + (n - 1 - 3d) = n + 2d - 1.$$ 

In general, $\sum_{e \in E(T)} g'(e) \geq n + d - 1$ and so $\text{val}_{\min}(T) \geq n + d - 1$. □
Next, we generalize Theorem 3.6 to caterpillars $T$ with $\Delta(T) = 3$ having an arbitrary number of vertices of degree 3.

**Theorem 4.2.** If $T$ is a caterpillar of order $n \geq 4$ such that $\Delta(T) = 3$ and $T$ has exactly $k$ vertices of degree 3, then

$$\text{val}_{\text{min}}(T) = n + k - 1.$$ 

**Proof.** Let $T$ be a caterpillar of order $n \geq 4$ with $\Delta(T) = 3$ such that $T$ contains $k$ vertices of degree 3. Then $\text{diam}(T) = n - k - 1$. Let $P : v_0, v_1, v_2, \cdots, v_{n-k-1}$ be a path of length $n - k - 1$ in $T$. Let $i_1, i_2, \cdots, i_k$ be integers such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n - k - 2$ and $\text{deg} v_{i_j} = 3$ for $1 \leq j \leq k$. Furthermore, let $f : V(T) \to \{0, 1, \cdots, n - 1\}$ be the $\gamma$-labeling of $T$ defined by

$$f(v_t) = \begin{cases} 
    d(v_t, v_0) & \text{if } t \leq i_1, \\
    d(v_t, v_0) + \max\{j : i_j < t\} & \text{otherwise}
\end{cases}$$

and

$$f(u_j) = 1 + f(v_{i_j}).$$

Since $\text{val}(f) = n + k - 1$, it follows that $\text{val}_{\text{min}}(T) \leq n + k - 1$.

Next, we show that $\text{val}_{\text{min}}(T) \geq n + k - 1$. Let

$$f : V(T) \to \{0, 1, 2, \cdots, n - 1\}$$

be an arbitrary $\gamma$-labeling of $T$ and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let $P$ be a $u - v$ path in $T$. The length of $P$ is at most $\text{diam}(T) = n - k - 1$. Also, by Lemma 3.3

$$\sum_{e \in E(P)} f'(e) \geq |f(u) - f(v)| = n - 1.$$ 

Since there are at least $k$ edges of $T$ not on $P$, it follows that

$$\text{val}(f) = \sum_{e \in E(T)} f'(e) \geq (n - 1) + k,$$

and so $\text{val}_{\text{min}}(T) \geq n + k - 1$. 

We now present a lower bound for the minimum value of a tree in terms of its order, maximum degree, and diameter.
Theorem 4.3. If $T$ is a tree of order $n \geq 4$, maximum degree $\Delta$, and diameter $d$, then

$$\text{val}_{\text{min}}(T) \geq \frac{8n + \Delta^2 - 6\Delta - 4d + \delta_\Delta}{4},$$

where

$$\delta_\Delta = \begin{cases} 
0 & \text{if } \Delta \text{ is even,} \\
1 & \text{if } \Delta \text{ is odd.}
\end{cases}$$

Furthermore, this bound is sharp for paths and stars.

Proof. Let $f$ be a $\gamma$-labeling of $T$ and let $u, v \in V(T)$ such that $f(u) = 0$ and $f(v) = n - 1$. Let $P$ be a $u-v$ path in $T$. Let $x$ be a vertex of $T$ with $\text{deg } x = \Delta$. We consider two cases.

Case 1. $\Delta = 2k$ for some integer $k \geq 1$. Since (1) at most two edges of $T$ incident with $x$ can be labeled by $i$ for each $i$ with $1 \leq i \leq (k - 1)$ and (2) the length of $P$ is at most $d$, it follows that

$$\text{val}(f) \geq (n - 1) + 2[1 + 2 + \cdots + (k - 1)] + [(n - 1 - d) - (2k - 2)]$$
$$= 2n + k^2 - 3k - d = 2n + \frac{\Delta^2}{4} - \frac{3\Delta}{2} - d$$
$$= \frac{8n + \Delta^2 - 6\Delta - 4d}{4}.$$ 

Case 2. $\Delta = 2k + 1$ for some integer $k \geq 1$. By the same reasoning used in Case 1,

$$\text{val}(f) \geq (n - 1) + 2[1 + 2 + \cdots + (k - 1)] + k + [(n - 1 - d) - (2k - 1)]$$
$$= 2n - 1 + k^2 - 2k - d = 2n + \frac{(\Delta - 1)^2}{4} - \Delta - d$$
$$= \frac{8n + \Delta^2 - 6\Delta - 4d + 1}{4}.$$ 

That this bound is sharp for paths and stars follows by Theorems B and C.
5. Connected Graphs of Order $n$ with Minimum Value $n + 1$

In Theorem 3.6, all connected graphs of order $n \geq 4$ having minimum value $n$ are characterized. In particular, if $T$ is a caterpillar of order $n \geq 4$ whose only vertex of degree exceeding 2 has degree 3, then $\text{val}_{\text{min}}(T) = n$. In this section, we characterize those connected graphs of order $n \geq 5$ having minimum value $n + 1$. First, we show that every caterpillar of order $n \geq 5$ whose unique vertex of degree exceeding 2 has degree 4 must have minimum value $n + 1$.

**Lemma 5.1.** Let $T$ be a caterpillar of order $n \geq 5$. If $T$ has a unique vertex $v$ with degree greater than 2 and $\text{deg } v = 4$, then $\text{val}_{\text{min}}(T) = n + 1$.

**Proof.** By Lemma 3.3, $\text{val}_{\text{min}}(T) \geq n + 1$. It remains to show that $\text{val}_{\text{min}}(T) \leq n + 1$. Suppose that $T$ is obtained from path $v_1, v_2, \ldots, v_{n-2}$ by adding the vertices $v_{n-1}$ and $v_n$ and joining each of $v_{n-1}$ and $v_n$ to a vertex $v_k$, where $2 \leq k \leq n - 3$. Thus $v_k$ is the only vertex of degree greater than 2 in $T$ and $\text{deg } v_k = 4$. Define a $\gamma$-labeling $f$ of $T$ by

$$f(v_i) = \begin{cases} 
  i - 1 & \text{if } 1 \leq i \leq k - 1, \\
  i & \text{if } i = k, \\
  i + 1 & \text{if } k + 1 \leq i \leq n - 2, \\
  k - 1 & \text{if } i = n - 1, \\
  k + 1 & \text{if } i = n.
\end{cases}$$

Since $\text{val}(f) = n + 1$, it follows that $\text{val}_{\text{min}}(T) \leq n + 1$. 

For a fixed integer $n$, let $\mathcal{T}_1$ be the set of caterpillars $T$ of order $n \geq 5$ such that $T$ has a unique vertex $v$ with degree greater than 2 and $\text{deg } v = 4$ (as described in Lemma 5.1), let $\mathcal{T}_2$ be the set of trees $T$ of order $n$ such that $T$ is a caterpillar of order $n \geq 6$ with $\Delta(T) = 3$ and $T$ has exactly two vertices of degree 3, and let $\mathcal{T}_3$ be the set of trees $T$ of order $n \geq 7$ such that $T$ has a unique vertex $v$ of degree greater than 2 and $\text{deg } v = 3$, where the distance between $v$ and a nearest end-vertex of $T$ is 2. By Lemma 5.1 and Theorems 4.1 and 4.2, we have the following.
Corollary 5.2. Let $T$ be a tree of order $n$. If $T \in T_1 \cup T_2 \cup T_3$, then $\text{val}_{\min}(T) = n + 1$.

Lemma 5.3. Each of the trees $F_1, F_2,$ and $F_3$ in Figure 3 of order $n = 9, 8, 8$, respectively, has minimum value $n + 2$, that is,

$$\text{val}_{\min}(F_1) = 11 \text{ and } \text{val}_{\min}(F_2) = \text{val}_{\min}(F_3) = 10.$$ 

Figure 3: The graphs $F_1, F_2,$ and $F_3$.

Proof. For each integer $i$ with $1 \leq i \leq 3$, a $\gamma$-labeling $f_i$ of $F_i$ is shown in Figure 3. Since $\text{val}(f_1) = 11$ and $\text{val}(f_2) = \text{val}(f_3) = 10$, it follows that $\text{val}_{\min}(F_1) \leq 11$, $\text{val}_{\min}(F_2) \leq 10$, and $\text{val}_{\min}(F_3) \leq 10$.

Next, we show that $\text{val}_{\min}(F_1) \geq 11$. Let $g$ be $\gamma$-min labeling of $F_1$ and let $u, v \in V(F_1)$ such that $g(u) = 0$ and $g(v) = 8$. Suppose that $P$ is a $u - v$ path in $F_1$. Then $\sum_{e \in E(P)} f'(e) \geq 8$ by Lemma 3.4. Since there are at least three edges of $F_1$ not in $P$, it follows that $\text{val}_{\min}(F_1) = \text{val}(g) \geq 8 + 3 = 11$. A similar argument shows that $\text{val}_{\min}(F_2) \geq 10$, and $\text{val}_{\min}(F_3) \geq 10$.

We now characterize all trees of order $n \geq 5$ whose minimum value is $n + 1$.

Theorem 5.4. Let $T$ be a tree of order $n \geq 5$. Then $\text{val}_{\min}(T) = n + 1$ if and only if $T \in T_1 \cup T_2 \cup T_3$.

Proof. By Corollary 5.2, if $T \in T_1 \cup T_2 \cup T_3$, then $\text{val}_{\min}(T) = n + 1$. It therefore remains to verify the converse. We begin by establishing the following three claims.

Claims. Let $T$ be a tree of order $n \geq 7$ such that $\text{val}_{\min}(T) = n + 1$ and $T \notin T_1 \cup T_2 \cup T_3$. Then:

(1) $3 \leq \Delta(T) \leq 4$.

(2) $T$ has at most two vertices of degree greater than 2.
(3) If \( v \) is a vertex of \( T \) with \( \deg v \geq 3 \), then the distance between \( v \) and a nearest end-vertex in \( T \) is at most 2.

**Proof of Claims.** Since \( \text{val}_{\text{min}}(T) = n + 1 \), it follows that \( T \) is not a path by Theorem B and so \( \Delta(T) \geq 3 \). If \( \Delta(T) \geq 5 \), then \( \text{val}_{\text{min}}(T) \geq (n - 1) + 2^2 = n + 3 \) by Lemma 3.3, a contradiction. Thus \( 3 \leq \Delta(T) \leq 4 \) and so Claim (1) holds.

Next we verify Claim (2). Suppose that \( T \) has \( k \geq 3 \) vertices of degree greater than 2. Then \( T \) contains a caterpillar \( T' \) of order \( n' \) as a subgraph with \( \Delta(T') = 3 \) such that \( T' \) has exactly three vertices of degree 3. By Theorem 4.2, \( \text{val}_{\text{min}}(T') = n' + 2 \). It then follows from Lemma 3.1 that

\[
\text{val}_{\text{min}}(T) \geq [(n - 1) - (n' - 1)] + \text{val}_{\text{min}}(T') \geq (n - n') + (n' + 2) = n + 2,
\]

a contradiction. Thus Claim (2) holds.

We now verify Claim (3). Let \( v \) be a vertex of \( T \) with \( \deg v \geq 3 \). If the distance between \( v \) and a nearest end-vertex in \( T \) is greater than 2, then \( T \) contains a subtree \( T'' \) of order \( n'' \) such that (a) \( \Delta(T'') = 3 \) and \( T'' \) has a unique vertex \( v \) of degree 3 and (b) the distance \( d \) between \( v \) and a nearest end-vertex in \( T'' \) is greater than 2. By Theorem 4.1,

\[
\text{val}_{\text{min}}(T'') = n' + d - 1 \geq n' + 2.
\]

Again, by Lemma 3.1,

\[
\text{val}_{\text{min}}(T) \geq [(n - 1) - (n' - 1)] + \text{val}_{\text{min}}(T') \geq (n - n') + (n' + 2) = n + 2,
\]

a contradiction. Thus Claim (3) holds. This completes the proof of the three claims.

We continue with the proof of the theorem. Assume, to the contrary, that there is a tree \( T \) of order \( n \geq 7 \) with \( \text{val}_{\text{min}}(T) = n + 1 \) such that \( T \notin T_1 \cup T_2 \cup T_3 \). By Claim (1), \( 3 \leq \Delta(T) \leq 4 \). We consider two cases, according to whether \( \Delta(T) = 3 \) or \( \Delta(T) = 4 \).

**Case 1.** \( \Delta(T) = 3 \). If \( T \) is a caterpillar, then \( T \) contains exactly two vertices of degree 3 by Theorem 4.2. However then, \( T \in T_2 \), a contradiction. Thus \( T \) is not a caterpillar. If \( T \) has exactly one vertex \( x \) of degree 3, then the distance between \( x \) and a nearest end-vertex of \( T \) is 2 by Theorem 4.1. However then, \( T \in T_3 \), again a contradiction. Thus \( T \) is not a caterpillar.
and $T$ contains exactly two vertices $u$ and $v$ of degree 3 by Claim (2). Furthermore, we may assume that the distance $d$ from $u$ to a nearest end-vertex of $T$ is 2 by Claim (3). We consider three subcases.

**Subcase 1.1.** $d(u, v) \geq 3$. Then $T$ contains two edge-disjoint subgraphs $H_1$ and $H_2$ such that $H_1$ is isomorphic to the graph $F$ in Lemma 3.5 and $H_2$ is isomorphic to $K_{1,3}$. Let $f$ be a $\gamma$-min labeling of $T$. Since $\text{val}_{\min}(H_1) = 8$ by Lemma 3.5 and $\text{val}_{\min}(H_2) = 4$ by Theorem C, it follows by Lemma 3.2 that

$$\text{val}_{\min}(T) \geq [(n - 1) - 6 - 3] + (8 + 4) = n + 2,$$

a contradiction.

**Subcase 1.2.** $d(u, v) = 2$. Then $T$ contains the graph $F_1$ of Lemma 5.3 as a subgraph. Since the size of $F_1$ is 8 and $\text{val}_{\min}(F_1) = 11$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n - 1) - 8] + 11 = n + 2$, which produces a contradiction.

**Subcase 1.3.** $d(u, v) = 1$. Then $T$ contains the graph $F_2$ of Lemma 5.3 as a subgraph. Since the size of $F_2$ is 7 and $\text{val}_{\min}(F_2) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n - 1) - 7] + 10 = n + 2$, a contradiction.

**Case 2.** $\Delta(T) = 4$. There are two subcases.

**Subcase 2.1.** $T$ has a unique vertex $v$ of degree exceeding 2. Then $\deg v = 4$. If $T$ is a caterpillar, then $T \in T_1$, a contradiction. Thus $T$ is not a caterpillar. However then, $T$ contains the graph $F_3$ of Lemma 5.3 as a subgraph. Since the size of $F_3$ is 7 and $\text{val}_{\min}(F_3) = 10$ by Lemma 5.3, it follows from Lemma 3.1 that $\text{val}_{\min}(T) \geq [(n - 1) - 7] + 10 = n + 2$, a contradiction.

**Subcase 2.2.** $T$ has two vertices $u$ and $v$ of degree exceeding 2. If $T$ is not a caterpillar, then $\text{val}_{\min}(T) \geq n + 2$ by the proofs of Subcases 1.1, 1.2, and 1.3 in Case 1, which is a contradiction. Thus we may assume that $T$ is a caterpillar and $\deg u = 4$. There are two subcases.

**Subcase 2.2.1.** $d(u, v) \geq 2$. Then $T$ contains two edge-disjoint subgraphs isomorphic to $K_{1,4}$ and $K_{1,3}$, respectively. Let $f$ be a $\gamma$-min labeling of $T$. 

Since $\text{val}_{\text{min}}(K_{1,4}) = 6$ and $\text{val}_{\text{min}}(K_{1,3}) = 4$ by Theorem C, it follows from Lemma 3.2 that $\text{val}_{\text{min}}(T) \geq [(n - 1) - 4 - 3] + 6 + 4 = n + 2$, a contradiction.

Subcase 2.2.2. $d(u, v) = 1$. Then $T$ contains the double star $S_{4,3}$ as a subgraph. Since the size of $S_{4,3}$ is 6 and $\text{val}_{\text{min}}(S_{4,3}) = 9$ by Proposition 2.1, it follows by Lemma 3.1 that $\text{val}_{\text{min}}(T) \geq [(n - 1) - 6] + 9 = n + 2$, a contradiction. ■

We next characterize all connected graphs $G$ of order $n$ for which $\text{val}_{\text{min}}(G) = n + 1$. First, we present two lemmas. Since the proofs are straightforward, we omit them.

**Lemma 5.5.** For the graph $H$ of Figure 4, $\text{val}_{\text{min}}(H) = 9$.

![Figure 4: The graph $H$ of Lemma 5.5.](image)

Let $\mathcal{F}$ be the set of all graphs of order $n \geq 3$ obtained from the path $v_1, v_2, \ldots, v_n$ by joining $v_i$ and $v_{i+2}$ for some $i$ with $1 \leq i \leq n - 2$.

**Lemma 5.6.** If $F \in \mathcal{F}$, then $\text{val}_{\text{min}}(F) = n + 1$.

**Theorem 5.7.** Let $G$ be a connected graph of order $n$. Then $\text{val}_{\text{min}}(G) = n + 1$ if and only if $G \in T_1 \cup T_2 \cup T_3 \cup \mathcal{F}$.

**Proof.** We have seen in Theorem 5.4 and Lemma 5.6 that if $G \in T_1 \cup T_2 \cup T_3 \cup \mathcal{F}$, then $\text{val}_{\text{min}}(G) = n + 1$. For the converse, let $G$ be a connected graph for which $\text{val}_{\text{min}}(G) = n + 1$ such that $G \notin T_1 \cup T_2 \cup T_3$. It then follows from Theorem 5.4 that $G$ is not a tree. Hence $G$ contains cycles. By Theorem B, $G$ contains exactly one cycle $C$ and so $G$ has size $n$. Suppose that $C$ is a $k$-cycle, where $k \geq 3$. Since $\text{val}_{\text{min}}(G) = 2k - 2$ by Theorem D, it follows by Lemma 3.1 that

$$\text{val}_{\text{min}}(G) \geq (n - k) + (2k - 2) = n + k - 2.$$
Since $val_{\text{min}}(G) = n + 1$, the cycle $C$ is a triangle. If $G$ contains the graph $H$ of Figure 4 as a subgraph, then by Lemmas 5.5 and 3.1,

$$val_{\text{min}}(G) \geq (n - 6) + val_{\text{min}}(H) = (n - 6) + 9 = n + 3,$$

which is impossible. Therefore, at least one vertex of $C$ has degree 2 in $G$. Furthermore, $G$ contains no vertex of degree 4 or more; for otherwise, $G$ contains $K_{1,4}$ as a subgraph and by Lemma 3.1 and Theorem C,

$$val_{\text{min}}(G) \geq (n - 4) + val_{\text{min}}(K_{1,4}) = (n - 4) + 6 = n + 2,$$

a contradiction. Also, observe that there cannot be a vertex of degree 3 that does not belong to $C$; for otherwise, $G$ contains edge-disjoint subgraphs $K_3$ and $K_{1,3}$ and by Lemma 3.2, Theorems C and D,

$$val_{\text{min}}(G) \geq (n - 3 - 3) + val_{\text{min}}(K_3) + val_{\text{min}}(K_{1,3})
= (n - 6) + 4 + 4 = n + 2,$$

which is impossible. This implies that $G \in \mathcal{F}$.

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**References**

