

A SURVEY ON THE CYCLIC COLORING AND ITS RELAXATIONS

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Abstract

A cyclic coloring of a plane graph is a vertex coloring such that any two vertices incident with the same face receive distinct colors. This type of coloring was introduced more than fifty years ago, and a lot of research in chromatic graph theory was sparked by it. This paper is a survey on the state of the art concerning the cyclic coloring and relaxations of this graph invariant.

Keywords: plane graph, edge coloring, vertex coloring.

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