

A SURVEY ON PACKING COLORINGS

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Abstract

If $S = (a_1, a_2, \dots)$ is a non-decreasing sequence of positive integers, then an S -packing coloring of a graph G is a partition of $V(G)$ into sets $X_1, X_2, \dots$ such that for each pair of distinct vertices in the set X_i , the distance between them is larger than a_i . If there exists an integer k such that $V(G) = X_1 \cup \dots \cup X_k$, then the partition is called an S -packing k -coloring. The S -packing chromatic number of G is the smallest k such that G admits an S -packing k -coloring. If $a_i = i$ for every i , then the terminology reduces to packing colorings and packing chromatic number. Since the introduction of these generalizations of the chromatic number in 2008 more than fifty papers followed. Here we survey the state of the art on the packing coloring, and its generalization, the S -packing coloring. We also list several conjectures and open problems.

Keywords: packing coloring, packing chromatic number, subcubic graph, S -packing chromatic number, computational complexity.

2010 Mathematics Subject Classification: 05C12, 05C15, 05C69, 05C70, 68Q25.

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Received 31 January 2020

Revised 8 April 2020

Accepted 8 April 2020