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BALANCEDNESS AND THE LEAST LAPLACIAN EIGENVALUE OF SOME COMPLEX UNIT GAIN GRAPHS

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Abstract

Let $\mathbb{T}_4 = \{\pm 1, \pm i\}$ be the subgroup of 4-th roots of unity inside $\mathbb{T}$, the multiplicative group of complex units. A complex unit gain graph Φ is a simple graph $\Gamma = (V(\Gamma) = \{v_1, \ldots, v_n\}, E(\Gamma))$ equipped with a map $\varphi : \vec{E}(\Gamma) \to \mathbb{T}$ defined on the set of oriented edges such that $\varphi(v_i v_j) = \varphi(v_j v_i)^{-1}$. The gain graph Φ is said to be balanced if for every cycle $C = v_{i_1} v_{i_2} \cdots v_{i_k} v_{i_1}$ we have $\varphi(v_{i_1} v_{i_2})\varphi(v_{i_2} v_{i_3}) \cdots \varphi(v_{i_k} v_{i_1}) = 1$.

It is known that Φ is balanced if and only if the least Laplacian eigenvalue $\lambda_n(\Phi)$ is 0. Here we show that, if Φ is unbalanced and $\varphi(\Phi) \subseteq \mathbb{T}_4$, the eigenvalue $\lambda_n(\Phi)$ measures how far is Φ from being balanced. More precisely, let $\nu(\Phi)$ (respectively, $\epsilon(\Phi)$) be the number of vertices (respectively, edges) to cancel in order to get a balanced gain subgraph. We show that

$$\lambda_n(\Phi) \leq \nu(\Phi) \leq \epsilon(\Phi).$$

We also analyze the case when $\lambda_n(\Phi) = \nu(\Phi)$. In fact, we identify the structural conditions on Φ that lead to such equality.

Keywords: gain graph, Laplacian eigenvalues, balanced graph, algebraic frustration.

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