*Discussiones Mathematicae*
*Graph Theory 41 (2021) 513–530*
*doi:10.7151/dmgt.2206*



# THE VERTEX-RAINBOW CONNECTION NUMBER OF SOME GRAPH OPERATIONS

Hengzhe Li, Yingbin Ma

*College of Mathematics and Information Science*
*Henan Normal University, Xinxiang 453007, P.R. China*

**e-mail:** lhz@htu.cn
mybxy521@163.com

AND

Xueliang Li

*Center for Combinatorics and LPMC-TJKLC*
*Nankai University, Tianjin 300071, China*

**e-mail:** lxl@nankai.edu.cn

**Abstract**

A path in an edge-colored (respectively vertex-colored) graph $G$ is *rainbow* (respectively *vertex-rainbow*) if no two edges (respectively internal vertices) of the path are colored the same. An edge-colored (respectively vertex-colored) graph $G$ is *rainbow connected* (respectively *vertex-rainbow connected*) if every two distinct vertices are connected by a rainbow (respectively vertex-rainbow) path. The *rainbow connection number* $rc(G)$ (respectively *vertex-rainbow connection number* $rvc(G)$) of $G$ is the smallest number of colors that are needed in order to make $G$ rainbow connected (respectively vertex-rainbow connected). In this paper, we show that for a connected graph $G$ and any edge $e = xy \in E(G)$, $rvc(G) \leq rvc(G-e) \leq rvc(G) + d_{G-e}(x,y) - 1$ if $G-e$ is connected. For any two connected, non-trivial graphs $G$ and $H$, $rad(G\Box H) - 1 \leq rvc(G\Box H) \leq 2rad(G\Box H)$, where $G\Box H$ is the Cartesian product of $G$ and $H$. For any two non-trivial graphs $G$ and $H$ such that $G$ is connected, $rvc(G \circ H) = 1$ if $diam(G \circ H) \leq 2$, $rad(G) - 1 \leq rvc(G \circ H) \leq 2rad(G)$ if $diam(G) > 2$, where $G \circ H$ is the lexicographic product of $G$ and $H$. For the line graph $L(G)$ of a graph $G$ we show that $rvc(L(G)) \leq rc(G)$, which is the first known nontrivial inequality between the rainbow connection number and vertex-rainbow connection number. Moreover, the bounds reported are tight or tight up to additive constants.

**Keywords:** rainbow connection number, vertex-rainbow connection number, Cartesian product, lexicographic product, line graph.

**2010 Mathematics Subject Classification:** 05C15, 05C40, 05C76.

## References


[1] M. Basavaraju, L.S. Chandran, D. Rajendraprasad and A. Ramaswamy, *Rainbow connection number and radius*, Graphs Combin. **30** (2014) 275–285. doi:10.1007/s00373-012-1267-7

[2] M. Basavaraju , L.S. Chandran, D. Rajendraprasad and A. Ramaswamy, *Rainbow connection number of graph power and graph products*, Graphs Combin. **30** (2014) 1363–1382. doi:10.1007/s00373-013-1355-3

[3] J.P. Bode and H. Harborth, *The minimum size of k-rainbow connected graphs of given order*, Discrete Math. **313** (2013) 1924–1928. doi:10.1016/j.disc.2012.07.023

[4] J.A. Bondy and U.S.R. Murty, Graph Theory (New York, Springer, 2008).

[5] Y. Caro, A. Lev, Y. Roditty, Zs. Tuza and R. Yuster, *On rainbow connection*, Electron. J. Combin. **15** (2008) #R57.

[6] S. Chakraborty, E. Fischer, A. Matsliah and R. Yuster, *Hardness and algorithms for rainbow connection*, J. Comb. Optim. **21** (2011) 330–347. doi:10.1007/s10878-009-9250-9

[7] L.S. Chandran, A. Das, D. Rajendraprasad and N.M. Varma, *Rainbow connection number and connected dominating sets*, J. Graph Theory **71** (2012) 206–218. doi:10.1002/jgt.20643

[8] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *Rainbow connection in graphs*, Math. Bohem. **133** (2008) 85–98.

[9] G. Chartrand, G.L. Johns, K.A. McKeon and P. Zhang, *The rainbow connectivity of a graph*, Networks **54** (2009) 75–81. doi:10.1002/net.20296

[10] L. Chen, X. Li and H. Lian, *Further hardness results on the rainbow vertex-connection number of graphs*, Theoret. Comput. Sci. **481** (2013) 18–23. doi:10.1016/j.tcs.2013.02.012

[11] L. Chen, X. Li and Y. Shi, *The complexity of determining the rainbow vertex-connection of a graph*, Theoret. Comput. Sci. **412** (2011) 4531–4535. doi:10.1016/j.tcs.2011.04.032

[12] J. Ekstein, P. Holub, T. Kaiser, M. Koch, S. Matos Camacho, Z. Ryjáček and I. Schiermeyer, *The rainbow connection number of* 2*-connected graphs*, Discrete Math. **313** (2013) 1884–1892. doi:10.1016/j.disc.2012.04.022

[13] T. Gologranc, G. Mekiš and I. Peterin, *Rainbow connection and graph products*, Graphs Combin. **30** (2014) 591–607. doi:10.1007/s00373-013-1295-y

[14] X. Huang, H. Li, X. Li and Y. Sun, *Oriented diameter and rainbow connection number of a graph*, Discrete Math. Theor. Comput. Sci. **16** (2014) 51–60.

[15] X. Huang, X. Li, Y. Shi, J. Yue and Y. Zhao, *Rainbow connections for outerplanar graphs with diameter* 2 *and* 3, Appl. Math. Comput. **242** (2014) 277–280. doi:10.1016/j.amc.2014.05.066

[16] I. Gutman, *Distance in Thorny graphs*, Publ. Inst. Math. (Beograd) (N.S.) **63** (1998) 31–36.

[17] S. Klavžar and G. Mekiš, *On the rainbow connection of Cartesian products and their subgraphs*, Discuss. Math. Graph Theory **32** (2012) 783–793. doi:10.7151/dmgt.1644

[18] M. Knor, L'. Niepel and L'. Šoltés, *Centers in line graphs*, Math. Slovaca **43** (1993) 11–20.

[19] M. Krivelevich and R. Yuster, *The rainbow connection of a graph is* (*at most*) *reciprocal to its minimum degree*, J. Graph Theory **63** (2010) 185–191. doi:10.1002/jgt.20418

[20] H. Li, X. Li and S. Liu, *Rainbow connection of graphs with diameter* 2, Discrete Math. **312** (2012) 1453–1457. doi:10.1016/j.disc.2012.01.009

[21] H. Li, X. Li, Y. Sun and Y. Zhao, *Note on minimally d-rainbow connected graphs*, Graphs Combin. **30** (2014) 949–955. doi:10.1007/s00373-013-1309-9

[22] H. Li, X. Li and Y. Sun, *Rainbow connection number of graphs with diameter* 3, Discuss. Math. Graph Theory **37** (2017) 141–154. doi:10.7151/dmgt.1920

[23] H. Li and Y. Ma, *Rainbow connection number and graph operations*, Discrete Appl. Math. **230** (2017) 91–99. doi:10.1016/j.dam.2017.06.004

[24] X. Li and S. Liu, *Tight upper bound of the rainbow vertex-connection number for* 2*-connected graphs*, Discrete Appl. Math. **173** (2014) 62–69. doi:10.1016/j.dam.2014.04.002

[25] X. Li, S. Liu, L.S. Chandran, R. Mathew and D. Rajendraprasad, *Rainbow connection number and connecivity*, Electron. J. Combin. **19** (2012) #P20.

[26] X. Li and Y. Shi, *Rainbow connection in* 3*-connected graphs*, Graphs Combin. **29** (2013) 1471–1475. doi:10.1007/s00373-012-1204-9

[27] X. Li and Y. Sun, *Note on the rainbow k-connectivity of regular complete bipartite graphs*, Ars Combin. **101** (2011) 513–518.

[28] X. Li and Y. Sun, Rainbow Connections of Graphs (New York, Springer, 2012). doi:10.1007/978-1-4614-3119-0

[29] H. Liu, Â. Mestre and T. Sousa, *Rainbow vertex k-connection in graphs*, Discrete Appl. Math. **161** (2013) 2549–2555. doi:10.1016/j.dam.2013.04.025

[30] A. Lo, *A note on the minimum size of k-rainbow-connected graphs*, Discrete Math. **331** (2014) 20–21.
doi:10.1016/j.disc.2014.04.024

[31] Z. Lu and Y. Ma, *Graphs with vertex rainbow connection number two*, Sci. China Math. **58** (2015) 1803–1810.
doi:10.1007/s11425-014-4905-0

[32] Y. Mao, F. Yanling, Z. Wang and C. Ye, *Rainbow vertex-connection and graph products*, Int. J. Comput. Math. **93** (2016) 1078–1092.
doi:10.1080/00207160.2015.1047356

[33] I. Schiermeyer, *On minimally rainbow k-connected graphs*, Discrete Appl. Math. **161** (2013) 702–705.
doi:10.1016/j.dam.2011.05.001

Received 4 July 2018
Revised 22 January 2019
Accepted 22 January 2019