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THE MINIMUM SIZE OF A GRAPH WITH GIVEN TREE CONNECTIVITY

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Abstract

For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an S-tree is a such subgraph T of G that is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be internally disjoint if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$, and edge-disjoint if $E(T_1) \cap E(T_2) = \emptyset$. The generalized local connectivity $\kappa_G(S)$ (generalized local edge-connectivity $\lambda_G(S)$, respectively) is the maximum number of internally disjoint (edge-disjoint, respectively) S-trees in G. For an integer k with $2 \leq k \leq n$, the generalized k-connectivity (generalized k-edge-connectivity, respectively) is defined as $\kappa_k(G) = \min\{\kappa_G(S) \mid S \subseteq V(G), |S| = k\}$ ($\lambda_k(G) = \min\{\lambda_G(S) \mid S \subseteq V(G), |S| = k\}$, respectively).

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Let $f(n,k,t)$ ($g(n,k,t)$, respectively) be the minimum size of a connected graph G with order n and $\kappa_k(G) = t$ ($\lambda_k(G) = t$, respectively), where $3 \leq k \leq n$ and $1 \leq t \leq n - \lceil \frac{k}{2} \rceil$. For general k and t, Li and Mao obtained a lower bound for $g(n,k,t)$ which is tight for the case $k = 3$. We show that the bound also holds for $f(n,k,t)$ and is tight for the case $k = 3$. When t is general, we obtain upper bounds of both $f(n,k,t)$ and $g(n,k,t)$ for $k \in \{3,4,5\}$, and all of these bounds can be attained. When k is general, we get an upper bound of $g(n,k,t)$ for $t \in \{1,2,3,4\}$ and an upper bound of $f(n,k,t)$ for $t \in \{1,2,3\}$. Moreover, both bounds can be attained.

Keywords: generalized connectivity, tree connectivity, generalized k-connectivity, generalized k-edge-connectivity, packing.

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