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ORIENTED INCIDENCE COLOURINGS OF DIGRAPHS

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Abstract

Brualdi and Quinn Massey [6] defined incidence colouring while studying the strong edge chromatic index of bipartite graphs. Here we introduce a similar concept for digraphs and define the oriented incidence chromatic number. Using digraph homomorphisms, we show that the oriented incidence chromatic number of a digraph is closely related to the chromatic number of the underlying simple graph. This motivates our study of the oriented incidence chromatic number of symmetric complete digraphs. We give upper and lower bounds for the oriented incidence chromatic number of these graphs, as well as digraphs arising from common graph constructions and decompositions. Additionally we construct, for all $k \geqslant 2$, a target digraph H_k for which oriented incidence k colouring is equivalent to homomorphism to H_k.

Keywords: digraph homomorpism, graph colouring, incidence colouring, computational complexity.

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Appendix

Colour Class	Vertex List
1	$(1,(7,1)),(3,(3,2)),(2,(4,2)),(3,(4,3)),(3,(3,5)),$ $(5,(4,5)),(5,(6,5)),(3,(3,7)),(2,(7,2)),(5,(7,5)),$ $(6,(7,6)),(1,(6,1)),(6,(6,4)),(6,(6,2))$
2	$(6,(1,6)),(7,(1,7)),(2,(3,2)),(4,(3,4)),(4,(4,1)),$ $(4,(4,2)),(2,(5,2)),(4,(4,5)),(6,(5,6)),(4,(4,7)),$ $(7,(3,7)),(7,(5,7)),(6,(6,7)),(6,(6,3))$
3	$(1,(1,2)),(1,(1,3)),(1,(1,4)),(1,(1,5)),(1,(1,6)),$ $(1,(1,7)),(3,(2,3)),(4,(2,4)),(5,(2,5)),(6,(2,6)),$ $(7,(2,7))$
4	$(1,(2,1)),(1,(3,1)),(1,(4,1)),(5,(3,5)),(5,(5,4)),$ $(5,(5,1)),(5,(5,2)),(7,(4,7)),(7,(7,2)),(7,(7,3)),$ $(6,(3,6)),(5,(5,6)),(6,(4,6)),(7,(7,6)),(7,(7,1))$
5	$(2,(2,1)),(1,(5,1)),(1,(6,1)),(1,(3,1)),(2,(2,4)),$ $(2,(2,5)),(3,(3,4)),(3,(5,3)),(4,(5,4)),(3,(3,6)),$ $(2,(6,2)),(7,(7,4)),(7,(7,5)),(4,(6,4)),(7,(6,7))$
6	$(2,(1,2)),(3,(1,3)),(4,(1,4)),(5,(1,5)),(2,(2,3)),$ $(4,(4,3)),(2,(2,6)),(2,(2,7)),(5,(5,3)),(3,(6,3)),$ $(3,(7,3)),(4,(7,4)),(5,(5,7)),(4,(4,6)),(5,(6,5))$

Table 1. An oriented incidence colouring $\overrightarrow{K}_7$ with six colours.

Colour Class	**Vertex List**
1	$(6,(6,8)),(1,(2,1)),(3,(3,1)),(3,(3,2)),(2,(2,4)),$ $(3,(3,4)),(4,(5,4)),(6,(2,6)),(6,(6,1)),(6,(6,4)),$ $(6,(6,5)),(4,(7,4)),(8,(5,8)),(3,(5,3)),(1,(7,1)),$ $(6,(7,6)),(3,(7,3)),(1,(5,1))$
2	$(8,(8,5)),(8,(8,6)),(2,(2,1)),(2,(2,3)),(8,(3,8)),$ $(1,(3,1)),(4,(4,1)),(4,(3,4)),(4,(4,5)),(4,(4,7)),$ $(6,(3,6)),(2,(2,6)),(4,(4,6)),(6,(5,6)),(2,(5,2)),$ $(7,(5,7)),(2,(2,7))$
3	$(1,(8,1)),(2,(8,2)),(3,(8,3)),(4,(8,4)),(5,(8,5)),$ $(6,(8,6)),(7,(8,7)),(1,(1,2)),(1,(1,3)),(1,(1,4)),$ $(1,(1,5)),(1,(1,6)),(7,(7,2)),(7,(7,4)),(7,(7,5)),$ $(7,(7,6)),(7,(7,3))$
4	$(1,(1,8)),(1,(1,7)),(2,(2,8)),(3,(3,8)),(2,(2,4)),$ $(3,(4,3)),(8,(4,8)),(1,(4,1)),(3,(3,6)),(1,(6,1)),$ $(2,(6,2)),(5,(7,5)),(2,(2,5)),(3,(3,5)),(5,(4,5)),$ $(5,(6,5)),(2,(7,2)),(3,(3,7)),(8,(6,8)),(8,(7,8))$
5	$(8,(1,8)),(4,(4,8)),(5,(5,8)),(7,(7,8)),(8,(2,8)),$ $(3,(1,3)),(4,(1,4)),(3,(2,3)),(4,(4,2)),(4,(4,3)),$ $(5,(2,5)),(5,(5,1)),(3,(6,3)),(4,(6,4)),(5,(5,3)),$ $(7,(2,7)),(7,(7,1)),(5,(5,6)),(7,(6,7))$
6	$(8,(8,1)),(8,(8,2)),(8,(8,3)),(8,(8,4)),(8,(8,7)),$ $(2,(1,2)),(2,(3,2)),(5,(1,5)),(6,(1,6)),(7,(1,7)),$ $(2,(4,2)),(5,(5,2)),(6,(6,2)),(6,(6,3)),(6,(4,6)),$ $(5,(3,5)),(7,(4,7)),(5,(5,4)),(5,(5,7)),(6,(6,7)),$ $(7,(3,7))$

Table 2. An oriented incidence colouring $\overrightarrow{K}_8$ with six colours.

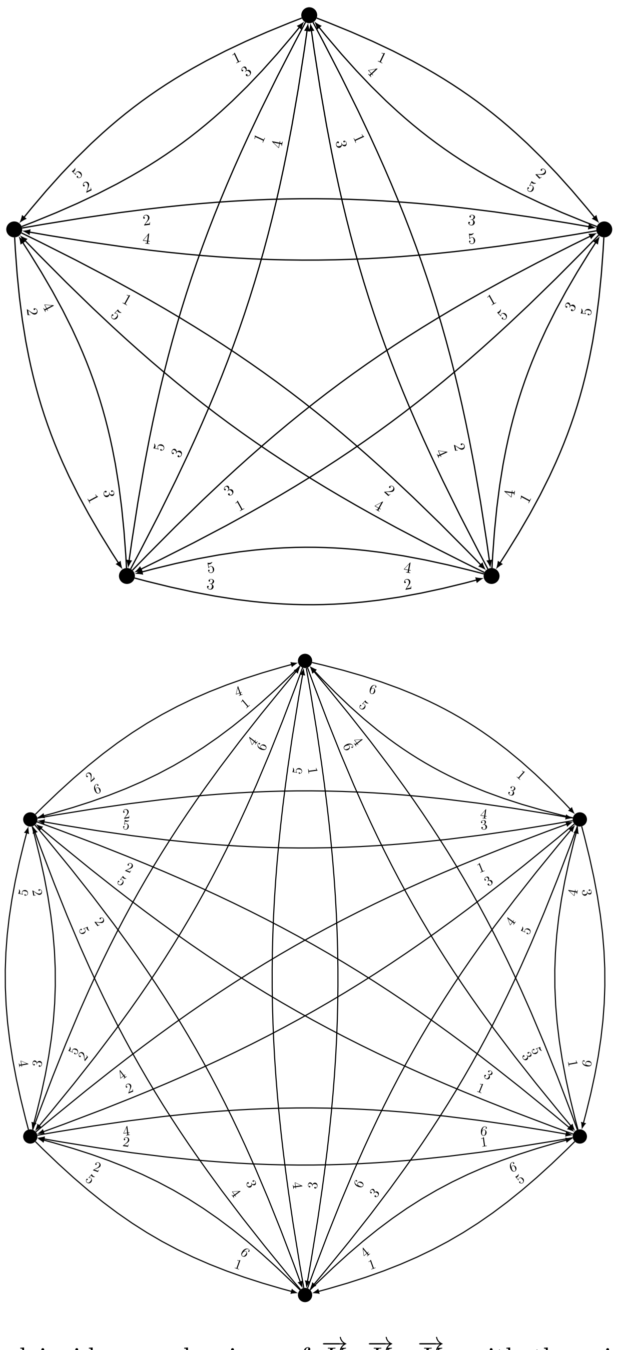

Figure 1. Oriented incidence colourings of $\overrightarrow{K}_4$, $\overrightarrow{K}_5$, $\overrightarrow{K}_6$ with the minimum number of colours. The colouring of $\overrightarrow{K}_4$ is obtained by deleting any vertex in the colouring of $\overrightarrow{K}_5$.