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# TURÁN FUNCTION AND $H$-DECOMPOSITION PROBLEM FOR GEM GRAPHS

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**Abstract**

Given a graph $H$, the *Turán function* $\mathrm{ex}(n, H)$ is the maximum number of edges in a graph on $n$ vertices not containing $H$ as a subgraph. For two graphs $G$ and $H$, an *$H$-decomposition* of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms a graph isomorphic to $H$. Let $\phi(n, H)$ be the smallest number $\phi$ such that any graph $G$ of order $n$ admits an $H$-decomposition with at most $\phi$ parts. Pikhurko and Sousa conjectured that $\phi(n, H) = \mathrm{ex}(n, H)$ for $\chi(H) \geq 3$ and all sufficiently large $n$. Their conjecture has been verified by Özkahya and Person for all edge-critical graphs $H$. In this article, we consider the *gem graphs* $\mathrm{gem}_4$ and $\mathrm{gem}_5$. The graph $\mathrm{gem}_4$ consists of the path $P_4$ with four vertices $a, b, c, d$ and edges $ab, bc, cd$ plus a universal vertex $u$ adjacent to $a, b, c, d$, and the graph $\mathrm{gem}_5$ is similarly defined with the path $P_5$ on five vertices. We determine

the Turán functions $\mathrm{ex}(n, \mathrm{gem}_4)$ and $\mathrm{ex}(n, \mathrm{gem}_5)$, and verify the conjecture of Pikhurko and Sousa when $H$ is the graph $\mathrm{gem}_4$ and $\mathrm{gem}_5$.

**Keywords:** gem graph, Turán function, extremal graph, graph decomposition.

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