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DESCRIBING NEIGHBORHOODS OF 5-VERTICES IN 3-POLYTOPES WITH MINIMUM DEGREE 5 AND WITHOUT VERTICES OF DEGREES FROM 7 TO 11[1]

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Abstract

In 1940, Lebesgue proved that every 3-polytope contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

$$(6,6,7,7,7),\ (6,6,6,7,9),\ (6,6,6,6,11),$$
$$(5,6,7,7,8),\ (5,6,6,7,12),\ (5,6,6,8,10),\ (5,6,6,6,17),$$
$$(5,5,7,7,13),\ (5,5,7,8,10),\ (5,5,6,7,27),$$
$$(5,5,6,6,\infty),\ (5,5,6,8,15),\ (5,5,6,9,11),$$
$$(5,5,5,7,41),\ (5,5,5,8,23),\ (5,5,5,9,17),$$
$$(5,5,5,10,14),\ (5,5,5,11,13).$$

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: $(5,5,6,6,\infty)$, $(5,6,6,6,15)$, $(6,6,6,6,6)$, where all parameters are tight.

Keywords: planar graph, structure properties, 3-polytope, neighborhood.

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