

DESCRIBING NEIGHBORHOODS OF 5-VERTICES
IN 3-POLYTOPES WITH MINIMUM DEGREE 5
AND WITHOUT VERTICES OF
DEGREES FROM 7 TO 11¹

OLEG V. BORODIN, ANNA O. IVANOVA

AND

OLESYA N. KAZAK

Institute of Mathematics Siberian Branch
Russian Academy of Sciences
Novosibirsk, 630090, Russia

e-mail: brdnoleg@math.nsc.ru
shmgnanna@mail.ru
agazandjelos@gmail.com

Abstract

In 1940, Lebesgue proved that every 3-polytope contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences:

(6, 6, 7, 7, 7), (6, 6, 6, 7, 9), (6, 6, 6, 6, 11),
(5, 6, 7, 7, 8), (5, 6, 6, 7, 12), (5, 6, 6, 8, 10), (5, 6, 6, 6, 17),
(5, 5, 7, 7, 13), (5, 5, 7, 8, 10), (5, 5, 6, 7, 27),
(5, 5, 6, 6, ∞), (5, 5, 6, 8, 15), (5, 5, 6, 9, 11),
(5, 5, 5, 7, 41), (5, 5, 5, 8, 23), (5, 5, 5, 9, 17),
(5, 5, 5, 10, 14), (5, 5, 5, 11, 13).

In this paper we prove that every 3-polytope without vertices of degree from 7 to 11 contains a 5-vertex for which the set of degrees of its neighbors is majorized by one of the following sequences: (5, 5, 6, 6, ∞), (5, 6, 6, 6, 15), (6, 6, 6, 6, 6), where all parameters are tight.

Keywords: planar graph, structure properties, 3-polytope, neighborhood.

2010 Mathematics Subject Classification: 05C15.

¹The work was funded by the Russian Science Foundation, grant 16–11–10054.

REFERENCES

- [1] V.A. Aksenov, O.V. Borodin and A.O. Ivanova, *Weight of 3-paths in sparse plane graphs*, Electron. J. Combin. **22** (2015) #P3.28.
- [2] K. Ando, S. Iwasaki and A. Kaneko, *Every 3-connected planar graph has a connected subgraph with small degree sum*, Annual Meeting of Mathematical Society of Japan (1993).
- [3] O.V. Borodin, *Solution of Kotzig's and Grünbaum's problems on the separability of a cycle in a planar graph*, Mat. Zametki **46** (1989) 9–12, in Russian.
- [4] O.V. Borodin, *Joint generalization of the theorems of Lebesgue and Kotzig on the combinatorics of planar maps*, Diskret. Mat. **3** (1991) 24–27, in Russian.
- [5] O.V. Borodin, *Joint extension of two Kotzig's theorems on 3-polytopes*, Combinatorica **13** (1993) 121–125.
doi:10.1007/BF01202794
- [6] O.V. Borodin, *Minimal vertex degree sum of a 3-path in plane maps*, Discuss. Math. Graph Theory **17** (1997) 279–284.
doi:10.7151/dmgt.1055
- [7] O.V. Borodin, *Colorings of plane graphs: a survey*, Discrete Math. **313** (2013) 517–539.
doi:10.1016/j.disc.2012.11.011
- [8] O.V. Borodin and A.O. Ivanova, *Describing $(d - 2)$ -stars at d -vertices, $d \leq 5$, in normal plane maps*, Discrete Math. **313** (2013) 1700–1709.
doi:10.1016/j.disc.2013.04.026
- [9] O.V. Borodin and A.O. Ivanova, *Describing 4-stars at 5-vertices in normal plane maps with minimum degree 5*, Discrete Math. **313** (2013) 1710–1714.
doi:10.1016/j.disc.2013.04.025
- [10] O.V. Borodin, A.O. Ivanova and T.R. Jensen, *5-stars of low weight in normal plane maps with minimum degree 5*, Discuss. Math. Graph Theory **34** (2014) 539–546.
doi:10.7151/dmgt.1748
- [11] O.V. Borodin and A.O. Ivanova, *An analogue of Franklin's Theorem*, Discrete Math. **339** (2016) 2553–2556.
doi:10.1016/j.disc.2016.04.019
- [12] O.V. Borodin and A.O. Ivanova, *Light and low 5-stars in normal plane maps with minimum degree 5*, Sibirsk. Mat. Zh. **57** (2016) 596–602, in Russian.
doi:10.1134/S0037446616030071
- [13] O.V. Borodin and D.R. Woodall, *Short cycles of low weight in normal plane maps with minimum degree 5*, Discuss. Math. Graph Theory **18** (1998) 159–164.
doi:10.7151/dmgt.1071
- [14] B. Ferencová and T. Madaras, *Light graph in families of polyhedral graphs with prescribed minimum degree, face size, edge and dual edge weight*, Discrete Math. **310** (2010) 1661–1675.
doi:10.1016/j.disc.2009.11.027

- [15] Ph. Franklin, *The four colour problem*, Amer. J. Math. **44** (1922) 225–236.
doi:10.2307/2370527
- [16] J. Harant and S. Jendrol', *On the existence of specific stars in planar graphs*, Graphs Combin. **23** (2007) 529–543.
doi:10.1007/s00373-007-0747-7
- [17] A.O. Ivanova and D.V. Nikiforov, *The structure of neighborhoods of 5-vertices in plane triangulation with minimum degree 5*, Mat. Zam. YaSU **20(2)** (2013) 66–78, in Russian.
- [18] A.O. Ivanova and D.V. Nikiforov, *Combinatorial structure of triangulated 3-polytopes with minimum degree 5*, in: Proceedings of the Scientific Conference of students, graduate students, and young researchers. XVII and XVIII Lavrent'ev's reading, Yakutsk; Kirov: International Center for Research Project (2015) 22–27, in Russian.
- [19] S. Jendrol', *A structural property of convex 3-polytopes*, Geom. Dedicata **68** (1997) 91–99.
doi:10.1023/A:1004993723280
- [20] S. Jendrol', *Paths with restricted degrees of their vertices in planar graphs*, Czechoslovak Math. J. **49** (1999) 481–490.
doi:10.1023/A:1022411100562
- [21] S. Jendrol' and M. Maceková, *Describing short paths in plane graphs of girth at least 5*, Discrete Math. **338** (2015) 149–158.
doi:10.1016/j.disc.2014.09.014
- [22] S. Jendrol', M. Maceková and R. Soták, *Note on 3-paths in plane graphs of girth 4*, Discrete Math. **338** (2015) 1643–1648.
doi:10.1016/j.disc.2015.04.011
- [23] S. Jendrol' and T. Madaras, *On light subgraphs in plane graphs of minimal degree five*, Discuss. Math. Graph Theory **16** (1996) 207–217.
doi:10.7151/dmgt.1035
- [24] S. Jendrol' and T. Madaras, *Note on an existence of small degree vertices with at most one big degree neighbour in planar graphs*, Tatra Mt. Math. Publ. **30** (2005) 149–153.
- [25] S. Jendrol' and H.-J. Voss, *Light subgraphs of graphs embedded in the plane—a survey*, Discrete Math. **313** (2013) 406–421.
doi:10.1016/j.disc.2012.11.007
- [26] A. Kotzig, *Contribution to the theory of Eulerian polyhedra*, Mat. Eas. SAV (Math. Slovaca) **5** (1955) 101–113.
- [27] A. Kotzig, *From the theory of Eulerian polyhedra*, Mat. Eas. SAV (Math. Slovaca) **13** (1963) 20–34, in Russian.
- [28] H. Lebesgue, *Quelques conséquences simples de la formule d'Euler*, J. Math. Pures Appl. **19** (1940) 27–43.

- [29] D.V. Nikiforov, *The structure of neighborhoods of 5-vertices in normal plane maps with minimum degree 5*, Mat. Zam. YaSU, **23** (2016) 56–66, in Russian.
- [30] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, in: Recent Progress in Combinatorics, W.T. Tutte, (Ed.), (Academic Press, New York, 1969) 287–293.
- [31] E. Steinitz, *Polyeder und Raumeinteilungen*, Enzykl. Math. Wiss. (Geometrie), 3AB **12** (1922) 1–139.
- [32] P. Wernicke, *Über den kartographischen Vierfarbensatz*, Math. Ann. **58** (1904) 413–426.
doi:10.1007/BF01444968

Received 12 July 2016
 Revised 13 January 2017
 Accepted 13 January 2017