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CHROMATIC PROPERTIES OF THE PANCAKE GRAPHS

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Abstract

Chromatic properties of the Pancake graphs $P_n, n \geqslant 2$, that are Cayley graphs on the symmetric group Sym_n generated by prefix-reversals are investigated in the paper. It is proved that for any $n \geqslant 3$ the total chromatic number of P_n is n, and it is shown that the chromatic index of P_n is $n-1$. We present upper bounds on the chromatic number of the Pancake graphs P_n, which improve Brooks' bound for $n \geqslant 7$ and Catlin's bound for $n \leqslant 28$. Algorithms of a total n-coloring and a proper $(n-1)$-coloring are given.

Keywords: Pancake graph, Cayley graphs, symmetric group, chromatic number, total chromatic number.

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1. Introduction

The Pancake graph $P_n = (Sym_n, PR), n \geqslant 2$, is the Cayley graph on the symmetric group Sym_n of permutations $\pi = [\pi_1\pi_2 \cdots \pi_n]$ written as strings in one-line notation, where $\pi_i = \pi(i)$ for any $1 \leqslant i \leqslant n$, with the generating set $PR = \{r_i \in Sym_n : 2 \leqslant i \leqslant n\}$ of all prefix-reversals r_i inversing the order of any substring $[1, i], 2 \leqslant i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1 \cdots \pi_i\pi_{i+1} \cdots \pi_n]r_i = [\pi_i \cdots \pi_1\pi_{i+1} \cdots \pi_n]$.

It is a connected vertex-transitive $(n-1)$-regular graph without loops and multiple edges of order $n!$. The graph P_n is almost pancyclic [8, 11] since it contains all cycles C_l of length l, where $6 \leqslant l \leqslant n!$, but does not contain cycles of length $3, 4$ or 5. This graph is well known because of the open combinatorial Pancake problem of finding its diameter [6].

A mapping $c : V(\Gamma) \to \{1, 2, \ldots, k\}$ is called a *proper* k*-coloring* of a graph $\Gamma = (V, E)$ if $c(u) \neq c(v)$ whenever the vertices u and v are adjacent. The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the least number of colors needed to color vertices of Γ. A subset of vertices assigned to the same color forms an independent set, i.e., a k-coloring is the same as a partition of the vertex set into k independent sets. The *chromatic index* $\chi'(\Gamma)$ of a graph Γ is the least number of colors needed to color edges of Γ such that no two adjacent edges share the same color. In the *total coloring* of a graph Γ it is assumed that no adjacent vertices, no adjacent edges, and no edge and its endvertices are assigned the same color. The *total chromatic number* $\chi''(\Gamma)$ of a graph Γ is the least number of colors needed in any total coloring of Γ.

In this paper we prove the following result on the total chromatic number.

Theorem 1. *$\chi''(P_n) = n$ for any $n \geqslant 3$.*

The chromatic index of the Pancake graphs is obtained from Vizing's bound $\chi' \geqslant \Delta$ [13] taking into account the edge coloring, in which the color $(i-1)$ is assigned to the prefix-reversal r_i, $2 \leqslant i \leqslant n$:

$$\chi'(P_n) = n - 1 \text{ for any } n \geqslant 3.$$

Since P_n, $n \geqslant 2$, is an $(n-1)$-regular graph, then by Brooks' theorem [2], stating that $\chi(\Gamma) \leqslant \Delta(\Gamma)$ for any connected graph Γ with the maximum degree $\Delta = \Delta(\Gamma)$, except for two cases, complete graphs and odd cycles, we have $\chi(P_n) \leqslant n-1$ for any $n \geqslant 3$. Let us note that $\chi(P_3) = 2$ since $P_3 \cong C_6$, and $\chi(P_4) = 3$ since there are 7-cycles in P_n, $n \geqslant 4$.

Thus, the trivial lower and upper bounds on the chromatic number of the Pancake graphs are given as follows:

$$3 \leqslant \chi(P_n) \leqslant n - 1 \text{ for any } n \geqslant 4. \tag{1}$$

Borodin and Kostochka [1] showed that the Brooks' bound is improved by 1 for graphs with $\omega \leqslant (\Delta - 1)/2$, where ω is the size of the maximum clique in the graph. The same result was obtained independently by Catlin [3]. Since $\omega(P_n) = 2$, then $\chi(P_n) \leqslant n - 2$ for any $n \geqslant 6$. Moreover, there is a proper 3-coloring of P_5. An example of such a proper 3-coloring for P_5 is presented in Appendix, where vertices $[\pi_1\pi_2\cdots\pi_{n-1}\pi_n]$ and $[\pi_n\pi_{n-1}\cdots\pi_2\pi_1]$ are assumed to be connected by an edge corresponding to r_n. Thus, finally we have

$$\chi(P_n) \leqslant n - 2 \text{ for any } n \geqslant 5. \tag{2}$$

The Catlin's bound for C_4-free graphs [4], that is $\chi \leqslant \frac{2}{3}(\Delta + 3)$, gives us $\chi(P_n) \leqslant \frac{2}{3}(n + 2)$ for any $n \geqslant 8$. The asymptotic bound $\chi(P_n) \leqslant O\left(\frac{n-1}{\log(n-1)}\right)$ follows from the Johansson result [7] for triangle-free graphs.

The following bounds on the chromatic number of the Pancake graphs are obtained in the paper.

Theorem 2. *The following hold for P_n:*

(i) *if* $5 \leqslant n \leqslant 8$, *then*

$$\chi(P_n) \leqslant \begin{cases} n-k, & \text{if } n \equiv k \pmod 4 \text{ for } k=1,3; \\ n-2, & \text{if } n \text{ is even;} \end{cases} \tag{3}$$

(ii) *if* $9 \leqslant n \leqslant 16$, *then*

$$\chi(P_n) \leqslant \begin{cases} n-(k+2), & \text{if } n \equiv k \pmod 4 \text{ for } k=1,3; \\ n-4, & \text{if } n \text{ is even;} \end{cases} \tag{4}$$

(iii) *if* $n \geqslant 17$, *then*

$$\chi(P_n) \leqslant \begin{cases} n-(k+4), & \text{if } n \equiv k \pmod 4 \text{ for } k=1,2,3; \\ n-8, & \text{if } n \equiv 0 \pmod 4). \end{cases} \tag{5}$$

The bounds obtained improve (2) for $n \geqslant 7$ and Catlin's bound for $n \leqslant 28$. The paper is organized as follows. Theorem 1 is proved and an algorithm of total n-coloring is presented in Section 2. An algorithm of proper $(n-1)$-coloring of P_n, $n \geqslant 3$, is given in Section 3 and Theorem 2 is proved in Section 4.

2. Total n-Coloring

The total coloring of a graph assumes that no adjacent vertices, no adjacent edges, and no edge and its endvertices are assigned the same color. The least number of colors needed in any total coloring of a graph Γ gives its total chromatic number $\chi''(\Gamma)$. It is evident that $\chi''(\Gamma) \geqslant \Delta(\Gamma)+1$, so $\chi''(P_n) \geqslant n$.

The main result of this section is a proof of existence of a total n-coloring of $P_n, n \geqslant 3$, which is based on efficient dominating sets in the graph.

An independent set D of vertices in a graph Γ is an *efficient dominating set* if each vertex not in D is adjacent to exactly one vertex in D. There are n efficient dominating sets in P_n [5, 9, 12] given by

$$D_k = \{[k\,\pi_2 \cdots \pi_n],\ \pi_j \in \{1,\ldots,n\}\backslash\{k\},\ 2 \leqslant j \leqslant n\},\ 1 \leqslant k \leqslant n. \tag{6}$$

From (6) it follows that $|D_{k_1} \bigcap D_{k_2}| = 0$, where $1 \leqslant k_1 \leqslant n$, $1 \leqslant k_2 \leqslant n$, $k_1 \neq k_2$, hence $n!$ vertices of P_n are colored by n colors such that $c(\pi) = k$ for any $\pi \in D_k$, $1 \leqslant k \leqslant n$.

The total n-coloring of P_n is based on its hierarchical structure, which can be presented as follows. For any $n \geqslant 3$ the graph P_n is constructed from n copies of $P_{n-1}(i), 1 \leqslant i \leqslant n$, where each $P_{n-1}(i)$ has the vertex set $V_i = \{[\pi_1 \cdots \pi_{n-1} i]$, where $\pi_k \in \{1,\ldots,n\}\backslash\{i\} : 1 \leqslant k \leqslant n-1\}$, $|V_i| = (n-1)!$, and the edge set $E_i = \{\{[\pi_1 \cdots \pi_{n-1} i], [\pi_1 \cdots \pi_{n-1} i]\, r_j\} \; : \; 2 \leqslant j \leqslant n-1\}$, $|E_i| = \frac{(n-1)!(n-2)}{2}$. Any two copies $P_{n-1}(i), P_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges $\{[i\pi_2 \cdots \pi_{n-1} j],\, [j\pi_{n-1} \cdots \pi_2 i]\}$, where

$$[i\pi_2 \cdots \pi_{n-1} j] r_n = [j\pi_{n-1} \cdots \pi_2 i]. \tag{7}$$

Prefix-reversals r_j, for $2 \leqslant j \leqslant n-1$, define *internal edges* in all n copies $P_{n-1}(i)$, and the prefix-reversal r_n defines *external edges* E_{ex} between copies. Copies $P_{n-1}(i)$ are also called $(n-1)$-*copies*.

From (6) and from the hierarchical structure of P_n it follows that efficient dominating sets of copies $P_{n-1}(i)$ are presented by all permutations with the first and the last elements fixed, namely

$$D_k^i = \{[k\, \pi_2\, \cdots\, \pi_{n-1}\, i],\, \pi_j \in \{1, \ldots, n\} \backslash \{k,\, i\},\, 2 \leqslant j \leqslant n\}, \tag{8}$$

where $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant n$, $k \neq i$. There are $n\,(n-1)$ such sets of cardinality $(n-2)!$. The sets defined by (6) and (8) admit the following obvious relationship:

$$D_k = \bigcup_{i=1, i\neq k}^{n} D_k^i \quad \text{for any} \quad k = 1, \ldots, n. \tag{9}$$

Let us consider the following sets of edges:

$$E_{i,j} = \{\{\pi, \tau\} \in E(P_n) : \, \pi \in D_i, \tau \in D_j\}\,, \tag{10}$$

where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $i \neq j$,

$$E_{i,j}^k = \left\{\{\pi, \tau\} \in E(P_n) : \, \pi \in D_i^k, \tau \in D_j^k\right\}, \tag{11}$$

where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$, $i \neq j$, $i \neq k$, $j \neq k$, and D_i, D_j are presented by (6), and D_i^k, D_j^k are presented by (8). Since P_n is an undirected graph, $E_{i,j} = E_{j,i}$. Sets given by (10) and (11) have independent edges, however sets E_{i_1,j_1} and E_{i_2,j_2} as well as sets E_{i_1,j_1}^k and E_{i_2,j_2}^k may have adjacent edges in P_n if and only if $|\{i_1, j_1\} \cap \{i_2, j_2\}| = 1$. We use this fact when colors are assigned to edges in P_n.

We assign colors for vertices and edges of P_n from the set $\{0, \ldots, n-1\}$ as follows:

(a) if n is odd then

$$\begin{cases} c(D_k) = 2k \,(\mathrm{mod}\ \ n), \\ c(E_{i,j}) = i + j \,(\mathrm{mod}\ \ n), \end{cases} \tag{12}$$

where $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n$, $i \neq k$, $j \neq k$;

(b) if n is even then

$$\begin{cases} c(D_k^\ell) = 2(k - |k > \ell| + \ell) \pmod{(n-1)}, \\ c(E_{i,j}^\ell) = i + j - |i > \ell| - |j > \ell| + 2\ell \pmod{(n-1)}, \\ c(E_{ex}) = n - 1, \end{cases} \tag{13}$$

where $1 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n$, $1 \leqslant \ell \leqslant n$, $i \neq \ell$, $j \neq \ell$, $k \neq \ell$, and $|p| = 1$, if p is truly, and $|p| = 0$, otherwise.

***Proof of Theorem* 1.** Let us show that (12) and (13) give a total n-coloring of P_n for any $n \geqslant 3$. We give a proof by a contradiction, assuming that such a coloring is not a total coloring, which means that some adjacent vertices, or adjacent edges, or an edge and its endvertices are assigned the same color.

If n is odd, there exist i, j, k, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$, $i \neq j$, $i \neq k$, $j \neq k$, such that one of the following equalities holds:

$$c(D_i) = c(D_j),\ c(D_i) = c(E_{i,j}),\ c(E_{i,j}) = c(E_{i,k}).$$

Let us show that this is not true. Indeed, if there exist i, j, $i \neq j$, such that $c(D_i) = c(D_j)$, then $2i \equiv 2j \pmod{n}$ by (12). Since n is odd, this is possible only when $i = j$, which contradicts to the initial conditions. If $c(D_i) = c(E_{i,j})$, then $2i \equiv i + j \pmod{n}$ by (12), which means that $i = j$ and we again have a contradiction. If $c(E_{i,j}) = c(E_{i,k})$, then $i + j \equiv i + k \pmod{n}$ by (12) and we have $j = k$, but this is also not true. Hence, our assumption was wrong, and an n-coloring of the graph P_n given by (12) is total.

If n is even, then first we show that each of copies $P_{n-1}(\ell)$, $1 \leqslant \ell \leqslant n$, of the graph P_n, $n \geqslant 3$, has a total coloring given by (13). Let us assume that this is not true, i.e., there exist i, j, k, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant k \leqslant n$, $i \neq j$, $i \neq k$, $i \neq \ell$, $j \neq k$, $j \neq \ell$, $k \neq \ell$, such that one of the following equalities holds:

$$c(D_i^\ell) = c(D_j^\ell),\ c(D_i^\ell) = c(E_{i,j}^\ell),\ c(E_{i,j}^\ell) = c(E_{i,k}^\ell).$$

Let us check the first equality. If there exist i, j, ℓ, where $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, $1 \leqslant \ell \leqslant n$, $i \neq j$, $i \neq \ell$, $j \neq \ell$, such that $c(D_i^\ell) = c(D_j^\ell)$, then $i - |i > \ell| \equiv j - |j > \ell| \pmod{(n-1)}$, but this leads to a contradiction on the initial conditions. Indeed, if $i > j > \ell$ or $i < j < \ell$, then $i = j$, which is not true. If $j < \ell < i$, then $i - 1 = j$, which is also not true, since $i < j$. If $i < \ell < j$, then $i = j - 1$, which contradicts to $i + 1 < j$. The other two equalities are checked in the same way.

Thus, each of copies $P_{n-1}(\ell)$, $1 \leqslant \ell \leqslant n$, of the graph P_n has a total coloring.

The external edges E_{ex} of P_n are colored by (13) with a color $(n-1)$. These edges are incident to vertices from efficient dominating sets D_i^j and D_j^i having different colors. Assume that $c(D_i^j) = c(D_j^i)$ for some i, j, $i \neq j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$. Then $i - |i > j| + j \equiv j - |j > i| + i \pmod{(n-1)}$, and hence $0 \equiv 1 \pmod{(n-1)}$, which holds only for $n = 2$. Thus, formulas (12) and (13) give a total coloring of the graph P_n for any $n \geqslant 3$, and hence $\chi''(P_n) = n$. ∎

Table 1. Efficient dominating sets of $(n-1)$-copies of P_n

	$P_{n-1}(1)$	$P_{n-1}(2)$	$\dots$	$P_{n-1}(i)$	$\dots$	$P_{n-1}(j)$	$\dots$	$P_{n-1}(n)$
$P_{n-1}(1)$	-	D_2^1	$\dots$	D_i^1	$\dots$	D_j^1	$\dots$	D_n^1
$P_{n-1}(2)$	D_1^2	-	$\dots$	D_i^2	$\dots$	D_j^2	$\dots$	D_n^2
$\dots$	$\dots$	$\dots$	-	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$P_{n-1}(i)$	D_1^i	D_2^i	$\dots$	-	$\dots$	D_j^i	$\dots$	D_n^i
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	-	$\dots$	$\dots$	$\dots$
$P_{n-1}(j)$	D_1^j	D_2^j	$\dots$	D_i^j	$\dots$	-	$\dots$	D_n^j
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	-	$\dots$
$P_{n-1}(n)$	D_1^n	D_2^n	$\dots$	D_i^n	$\dots$	D_j^n	$\dots$	-

3. PROPER $(n-1)$-COLORING

Efficient dominating sets (6) and (8) give us a simple algorithm of a proper $(n-1)$-coloring for $n!$ vertices of P_n, $n \geqslant 3$. It follows from (7) and (8) that

$$D_j^i\, r_n = D_i^j, \tag{14}$$

for any $i \neq j$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, where $X r_n = Y$ means $x r_n = y$ for any $x \in X, y \in Y$, i.e., external edges between any two copies $P_{n-1}(i)$ and $P_{n-1}(j)$ of P_n are incident to vertices from the sets D_j^i and D_i^j of corresponding copies as indicated by their superscripts.

These relationships between $(n-1)$-copies of the graph are shown in Table 1, where the entry in the i-th row and j-th column is referred to as the set D_j^i of a copy $P_{n-1}(i)$ in P_n, whose vertices are adjacent to vertices of a copy $P_{n-1}(j)$ by the external edges. The union of all entries in the i-th row gives the vertex set of a copy $P_{n-1}(i)$, i.e., $\bigcup_{j=1, j\neq i}^{n} D_j^i = V(P_{n-1}(i))$, and the union of all entries in the j-th column gives the efficient dominating set D_j, since by (9) we have $\bigcup_{i=1, i\neq j}^{n} D_j^i = D_j$. Moreover, the union of all sets (6) gives the vertex set of P_n.

Thus, on the one hand we have

$$\bigcup_{i=1, j=1, i\neq j}^{n} D_j^i = V(P_n) \text{ for any } n \geqslant 3,$$

and on the other hand we have

$$\bigcap_{i=1, j=1, i\neq j}^{n} D_i^j = 0,$$

which means that a proper $(n-1)$-coloring of all sets (8) is a proper $(n-1)$-coloring for the graph at all. Below we present an algorithm for such a coloring. We write $c(X) = c$ when $c(x) = c$ for any $x \in X$.

Algorithm *PC*.
Step 1. Set $c(D_j^1) = j - 1$ for every $2 \leqslant j \leqslant n$.
Step 2. For every $2 \leqslant i \leqslant n$ and every $2 \leqslant j \leqslant n$, where $j \neq i$, set

$$c(D_j^i) = \begin{cases} c(D_{j+1}^{i-1}) + 1 \,(\mathrm{mod}\,(n-1)), & \text{if } i \neq j+2, \\ c(D_{j+2}^{i-2}) + 1 \,(\mathrm{mod}\,(n-1)), & \text{if } i = j+2. \end{cases} \tag{15}$$

Theorem 3. *Algorithm PC gives a proper $(n-1)$-coloring of P_n, for $n \geqslant 3$.*

Proof. The following properties correspond to a proper $(n-1)$-coloring of P_n:
(a) vertices of the sets D_j^i and D_i^j, $i \neq j$, given by relationship (14) must have different colors, i.e., $c(D_j^i) \neq c(D_i^j)$ for any $i, j \in \{1, \ldots, n\}$;
(b) vertices of any $(n-1)$-copy must have a proper $(n-1)$-coloring.

Let us show that Algorithm *PC* holds these conditions. Indeed, Step 1 assigns colors from 1 up to $(n-1)$ to all sets D_j^1, $2 \leqslant j \leqslant n$, and Step 2 assigns a color $c(D_1^j) = c(D_j^1) + 1 \,(\mathrm{mod}\,(n-1))$ for each set D_1^j by (15). Similarly, from (15) for any $i < j$ we get

$$c(D_i^j) = c(D_j^i) + 1 \,(\mathrm{mod}\,(n-1)),$$

i.e., vertices of any two different efficient dominating sets of $(n-1)$-copies connected by external edges have different colors, hence property a) holds.

The property (b) holds since by (15) for a given i, $2 \leqslant i \leqslant n$, sets D_j^i get different $(n-1)$ colors for different j, $2 \leqslant j \leqslant n$, $j \neq i$, i.e., vertices of any $(n-1)$-copy have a proper $(n-1)$-coloring. Let us note that vertices of the same set $D_j = \bigcup_{i=1, i\neq j}^{n} D_j^i$ may have the same color since they are at distance at least three from each other, by the definition. Thus, properties (a) and (b) hold, so Algorithm *PC* gives a proper $(n-1)$-coloring of the graph P_n. ∎

4. UPPER BOUNDS FOR $\chi(P_n)$

Upper bounds on the chromatic number of the Pancake graph in Theorem 2 are obtained using the following properties of prefix-reversals.

Lemma 4. *The prefix-reversal r_i, $2 \leqslant i \leqslant n$, is even if and only if either $i \equiv 0 \,(\mathrm{mod}\ 4)$, or $i \equiv 1 \,(\mathrm{mod}\ 4)$.*

Proof. Let us consider prefix-reversals r_{2k} and r_{2k+1}, where $1 \leqslant k \leqslant \frac{(n-1)}{2}$, as a product of k transpositions:

$$\begin{cases} r_{2k} = (1\ 2k)(2\ 2k-1)\cdots(k\ k+1); \\ r_{2k+1} = (1\ 2k+1)(2\ 2k)\cdots(k\ k+2). \end{cases} \tag{16}$$

From (16) we immediately conclude that prefix-reversal r_i is even if and only if either $i \equiv 0 \,(\mathrm{mod}\ 4)$, or $i \equiv 1 \,(\mathrm{mod}\ 4)$. ∎

Lemma 5. *There is no expression for any prefix-reversal with even indexing as a product of prefix-reversals with odd indexing.*

Proof. As one can see from (16), any prefix-reversal r_i, $2 \leqslant i \leqslant n$, indexing by even i, changes parity of each elements of a substring $[1, i]$, but this does not hold for prefix-reversals with odd indexing. ∎

Lemma 6. *The Pancake graph P_n, $n \geqslant 3$, has $\frac{n!}{\ell}$ independent even ℓ-cycles, where $6 \leqslant \ell \leqslant 2n$.*

Proof. It is evident that there are at most $\frac{n!}{\ell}$ independent l-cycles in a graph with $n!$ vertices, so we have to describe such cycles for $\ell = 2k$, where $k \geqslant 3$. Let us consider cycles presented as product of ℓ prefix-reversals:

$$C_\ell = r_k r_{k-1} \cdots r_k r_{k-1} = (r_k r_{k-1})^k, \tag{17}$$

and let us show that they correspond to the lemma's conditions. Indeed, if $\pi = [\pi_1\, X]$, where $X = \pi_2 \cdots \pi_n$, then $\pi\, r_\ell r_{\ell-1} = [\pi_1\, X]\, r_\ell r_{\ell-1} = [X\, \pi_1]$, and therefore, $\pi\, (r_\ell r_{\ell-1})^\ell = [\pi_1\, X]\, (r_\ell r_{\ell-1})^\ell = [\pi_1\, X]$. Moreover, since any vertex of P_n is incident to exactly one edge corresponding to r_i, for any $2 \leqslant i \leqslant n$, hence the vertex must belong to the only cycle (17) and this completes the proof. ∎

Let us note that Lemma 6 is a particular case of Lemma 1 from [10].

***Proof of Theorem* 2.** We consider the set E of all even prefix-reversals. From Lemma 4 we have $E = \{r_i,\, 2 \leqslant i \leqslant n : i \equiv 0 \,(\mathrm{mod}\, 4) \text{ or } i \equiv 1 \,(\mathrm{mod}\, 4)\}$. Then

$$|E| = \begin{cases} \frac{n-k}{2}, & \text{if } n \equiv k \,(\text{mod } 4) \text{ for } k = 1, 3 \\ \frac{n-2}{2}, & \text{if } n \text{ is even.} \end{cases} \tag{18}$$

Indeed, if n is even, then there are exactly $\frac{n}{2}$ elements $i \in \{1, \ldots, n\}$ such that either $i \equiv 0 \,(\text{mod } 4)$, or $i \equiv 1 \,(\text{mod } 4)$, hence there are exactly $\frac{n-2}{2}$ such elements i, when $i \in \{2, \ldots, n\}$. Other cases in (18) are proved in the same way.

Now let us consider E-induced subgraphs of P_n on either even or odd permutations with edge sets defined by the set E. These subgraphs form two sets $P_n[E]^e$ and $P_n[E]^o$, where $P_n[E]^e$ contains all E-induced subgraphs on even permutations, and $P_n[E]^o$ contains all E-induced subgraphs on odd permutations.

These sets have the following trivial properties.

1. All subgraphs from $P_n[E]^e$ and $P_n[E]^o$ are $|E|$-regular.
2. There are no edges between vertices of subgraphs of $P_n[E]^e$ ($P_n[E]^o$) in P_n.
3. Vertices of all subgraphs from $P_n[E]^e$ and $P_n[E]^o$ form the vertex set of P_n.
4. There are edges between vertices of subgraphs from $P_n[E]^e$ and $P_n[E]^o$ in P_n, moreover, these edges correspond to the odd prefix-reversals.

Then, by Brooks' theorem and property 1, we have $\chi(P_n[E]^e) \leqslant |E|$ and $\chi(P_n[E]^o) \leqslant |E|$, and by properties 2–4 we immediately have $\chi(P_n) \leqslant 2\,|E|$, where $|E| > 2$, because otherwise odd cycles may appear. Let us show that this does not happen. Indeed, we have $|E| = 2$ when $n = 5, 6, 7$, so there are only two even prefix-reversals, namely, r_4 and r_5. Then, by Lemma 6 there are $n!/10$ independent 10-cycles presented by $(r_4 r_5)^5$ in P_n (see (17)), and hence, odd cycles do not appear. This means that

$$\chi(P_n) \leqslant 2\,|E| \text{ for any } n \geqslant 5. \tag{19}$$

Bound (3) in Theorem 2 follows from (18) and (19).

Since $\omega(P_n) = 2$, Borodin-Kostochka's bound holds for $P_n[E]^e$ and $P_n[E]^o$ independently when $|E| \geqslant 4$, and hence it holds for $n \geqslant 9$. From this we immediately have $\chi(P_n) \leqslant 2\,(|E| - 1)$ for any $n \geqslant 9$, which gives (4) in Theorem 2.

Now let us consider the set $F = \{r_{4i+1},\ 5 \leqslant 4i + 1 \leqslant n\}$ of prefix-reversals with odd indexing whose cardinality is

$$|F| = \begin{cases} \frac{n-k}{4}, & \text{if } n \equiv k \,(\mathrm{mod}\ 4) \text{ for } k = 1, 2, 3 \\ \frac{n-4}{4}, & \text{if } n \equiv 0 \,(\mathrm{mod}\ 4). \end{cases}$$

Applying similar arguments and taking into account Lemma 5, we get $\chi(P_n) \leqslant 4\,|F|$ for any $n \geqslant 5$ and $\chi(P_n) \leqslant 4\,(|F| - 1)$ for any $n \geqslant 17$, which gives (5) in Theorem 2 and completes the proof. ■

5. Discussions: Exact Values of the Chromatic Number

Thus, we have $\chi(P_3) = 2$ since $P_3 \cong C_6$, and $\chi(P_4) = 3$ since there are 7-cycles in P_n, $n \geqslant 4$. An example of a proper 3-coloring of P_5 is presented in Appendix. An optimal 4-coloring of P_6 was computed by Tomaž Pisanski, University of Primorska, and Jernej Azarija, University of Ljubljana, so $\chi(P_6) = 4$. Moreover, since P_{n-1} is an induced subgraph of P_n, then $\chi(P_7)$ is at least 4, and from (3) we have $\chi(P_7) = 4$. If $n = 8$ then from (3) we have $4 \leqslant \chi(P_8) \leqslant 6$, and from (4) we have $4 \leqslant \chi(P_n) \leqslant 6$, where $9 \leqslant n \leqslant 16$. However, a proper 4-coloring in these cases is unknown.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\chi(P_n)$	2	3	3	4	4	4?	4?	4?	4?	4?	4?	4?	4?	4?

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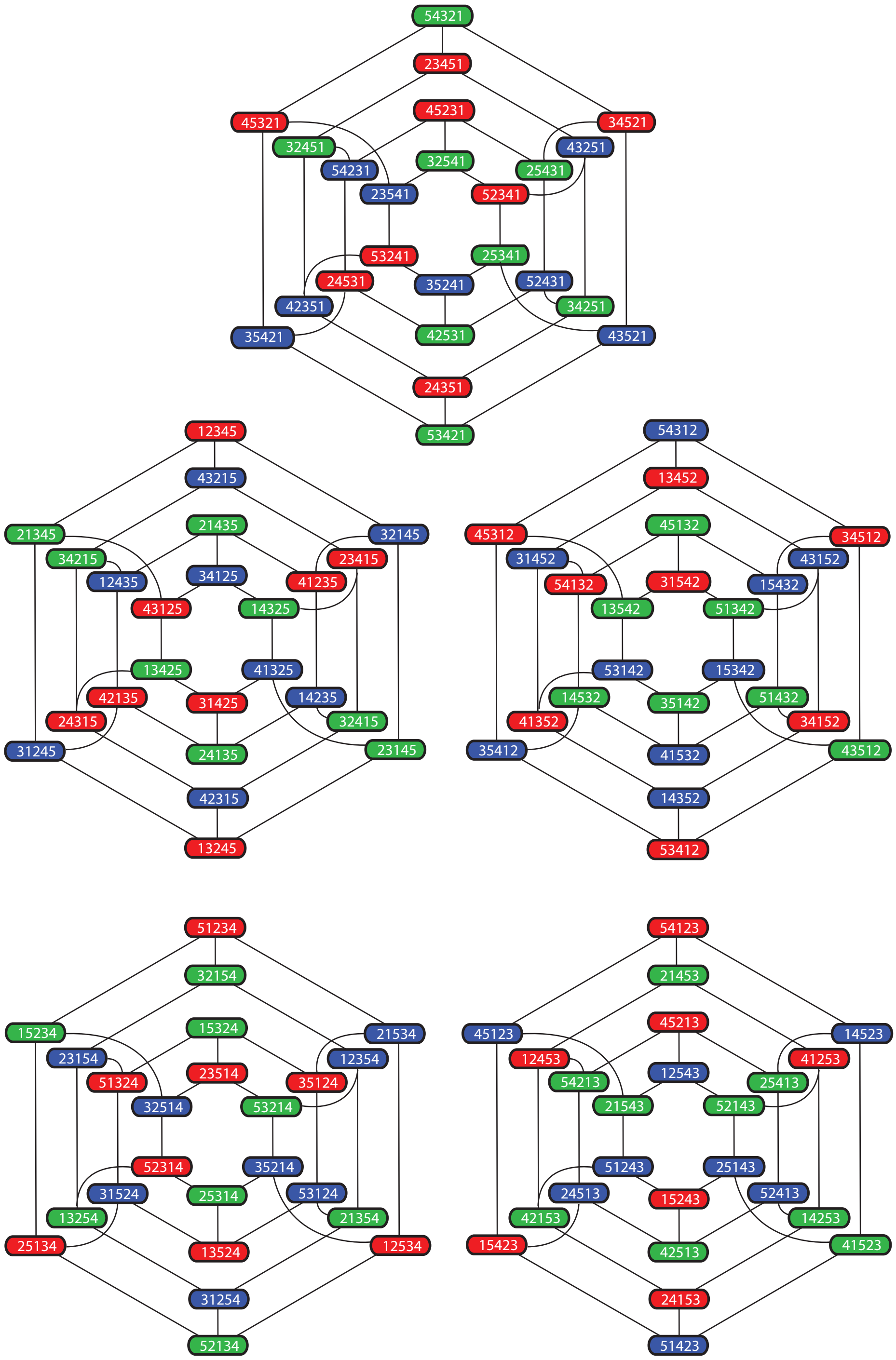

APPENDIX: Proper 3-coloring of P_5.

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