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CRITICALITY OF SWITCHING CLASSES OF REVERSIBLE 2-STRUCTURES LABELED BY AN ABELIAN GROUP

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Abstract

Let V be a finite vertex set and let $(\mathbb{A}, +)$ be a finite abelian group. An $\mathbb{A}$-labeled and reversible 2-structure defined on V is a function $g : (V \times V) \setminus \{(v, v) : v \in V\} \longrightarrow \mathbb{A}$ such that for distinct $u, v \in V$, $g(u, v) = -g(v, u)$. The set of $\mathbb{A}$-labeled and reversible 2-structures defined on V is denoted by $\mathscr{L}(V, \mathbb{A})$. Given $g \in \mathscr{L}(V, \mathbb{A})$, a subset X of V is a *clan* of g if for any $x, y \in X$ and $v \in V \setminus X$, $g(x, v) = g(y, v)$. For example, $\emptyset$, V and $\{v\}$ (for $v \in V$) are clans of g, called *trivial*. An element g of $\mathscr{L}(V, \mathbb{A})$ is *primitive* if $|V| \geq 3$ and all the clans of g are trivial.

The set of the functions from V to $\mathbb{A}$ is denoted by $\mathscr{S}(V, \mathbb{A})$. Given $g \in \mathscr{L}(V, \mathbb{A})$, with each $s \in \mathscr{S}(V, \mathbb{A})$ is associated the *switch* g^s of g by s defined as follows: given distinct $x, y \in V$, $g^s(x, y) = s(x) + g(x, y) - s(y)$. The *switching class* of g is $\{g^s : s \in \mathscr{S}(V, \mathbb{A})\}$. Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ and $X \subseteq V$, $\{g_{\restriction (X \times X) \setminus \{(x,x) : x \in X\}} : g \in \mathfrak{S}\}$ is a switching class, denoted by $\mathfrak{S}[X]$.

Given a switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, a subset X of V is a *clan* of $\mathfrak{S}$ if X is a clan of some $g \in \mathfrak{S}$. For instance, every $X \subseteq V$ such that $\min(|X|, |V \setminus X|) \leq 1$ is a clan of $\mathfrak{S}$, called *trivial*. A switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ is *primitive* if $|V| \geq 4$ and all the clans of $\mathfrak{S}$ are trivial. Given a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$, $\mathfrak{S}$ is *critical* if for each $v \in V$, $\mathfrak{S} - v$ is not primitive. First, we translate the main results on the primitivity of $\mathbb{A}$-labeled and reversible 2-structures in terms of switching classes. For instance, we prove the following. For a primitive switching class $\mathfrak{S} \subseteq \mathscr{L}(V, \mathbb{A})$ such that $|V| \geq 8$, there exist $u, v \in V$ such that $u \neq v$ and $\mathfrak{S}[V \setminus \{u, v\}]$ is primitive. Second, we characterize the critical switching classes by using some of the critical digraphs described in [Y. Boudabous and P. Ille, *Indecomposability graph and critical vertices of an indecomposable graph*, Discrete Math. **309** (2009) 2839–2846].

Keywords: labeled and reversible 2-structure, switching class, clan, primitivity, criticality.

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