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AN EXTENSION OF KOTZIG'S THEOREM

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Abstract

In 1955, Kotzig proved that every 3-connected planar graph has an edge with the degree sum of its end vertices at most 13, which is tight. An edge uv is of type (i, j) if $d(u) \leq i$ and $d(v) \leq j$. Borodin (1991) proved that every normal plane map contains an edge of one of the types $(3, 10)$, $(4, 7)$, or $(5, 6)$, which is tight. Cole, Kowalik, and Škrekovski (2007) deduced from this result by Borodin that Kotzig's bound of 13 is valid for all planar graphs with minimum degree δ at least 2 in which every d-vertex, $d \geq 12$, has at most $d - 11$ neighbors of degree 2.

We give a common extension of the three above results by proving for any integer $t \geq 1$ that every plane graph with $\delta \geq 2$ and no d-vertex, $d \geq 11 + t$, having more than $d - 11$ neighbors of degree 2 has an edge of one of the following types: $(2, 10 + t)$, $(3, 10)$, $(4, 7)$, or $(5, 6)$, where all parameters are tight.

Keywords: plane graph, normal plane map, structural property, weight.

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