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ON THE WEIGHT OF MINOR FACES IN TRIANGLE-FREE 3-POLYTOPES

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Abstract

The weight $w(f)$ of a face f in a 3-polytope is the degree-sum of vertices incident with f. It follows from Lebesgue's results of 1940 that every triangle-free 3-polytope without 4-faces incident with at least three 3-vertices has a 4-face with $w \leq 21$ or a 5-face with $w \leq 17$. Here, the bound 17 is sharp, but it was still unknown whether 21 is sharp.

The purpose of this paper is to improve this 21 to 20, which is best possible.

Keywords: plane map, plane graph, 3-polytope, structural property, weight of face.

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