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THE QUEST FOR A CHARACTERIZATION OF HOM-PROPERTIES OF FINITE CHARACTER

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Abstract

A *graph property* is a set of (countable) graphs. A *homomorphism* from a graph G to a graph H is an edge-preserving map from the vertex set of G into the vertex set of H; if such a map exists, we write $G \to H$. Given any graph H, the *hom-property* $\to H$ is the set of H*-colourable graphs*, i.e., the set of all graphs G satisfying $G \to H$. A graph property $\mathcal{P}$ is *of finite character* if, whenever we have that $F \in \mathcal{P}$ for every finite induced subgraph F of a graph G, then we have that $G \in \mathcal{P}$ too. We explore some of the relationships of the property attribute of being of finite character to other property attributes such as being *finitely-induced-hereditary*, being *finitely determined*, and being *axiomatizable*. We study the hom-properties of finite character, and prove some necessary and some sufficient conditions on H for $\to H$ to be of finite character. A notable (but known) sufficient condition is that H is a finite graph, and our new model-theoretic proof of this compactness result extends from hom-properties to all axiomatizable properties. In our quest to find an intrinsic characterization of those H for which $\to H$ is of finite character, we find an example of an infinite connected graph with no finite core and chromatic number 3 but with hom-property not of finite character.

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Keywords: (countable) graph, homomorphism (of graphs), property of graphs, hom-property, (finitely-)induced-hereditary property, finitely determined property, (weakly) finite character, axiomatizable property, compactness theorems, core, connectedness, chromatic number, clique number, independence number, dominating set.

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References

[1] B.L. Bauslaugh, *Core-like properties of infinite graphs and structures*, Discrete Math. **138** (1995) 101–111.
doi:10.1016/0012-365X(94)00191-K

[2] B.L. Bauslaugh, *Cores and compactness of infinite directed graphs*, J. Combin. Theory Ser. B **68** (1996) 255–276.

[3] B.L. Bauslaugh, *List-Compactness of directed graphs*, Graphs Combin. **17** (2001) 17–38.
doi:10.1007/s003730170052

[4] M. Borowiecki, I. Broere, M. Frick, G. Semanišin and P. Mihók, *A survey of hereditary properties of graphs*, Discuss. Math. Graph Theory **17** (1997) 5–50.
doi:10.7151/dmgt.1037

[5] J. Bucko and P. Mihók, *On infinite uniquely partitionable graphs and graph properties of finite character*, Discuss. Math. Graph Theory **29** (2009) 241–251.
doi:10.7151/dmgt.1444

[6] G. Chartrand, L. Lesniak and P. Zhang, Graphs and Digraphs, Fifth Edition (CRC Press, Boca Raton, 2011).

[7] R. Cowen, S.H. Hechler and P. Mihók, *Graph coloring compactness theorems*, Sci. Math. Jpn. **56** (2002) 171–180.

[8] B.A. Davey and H.A. Priestly, Introduction to Lattices and Order, Second Edition (Cambridge University Press, New York, 2008).

[9] N.G. de Bruijn and P. Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Indag. Math. **13** (1951) 371–373.
doi:10.1016/S1385-7258(51)50053-7

[10] R. Diestel, Graph Theory, Fourth Edition (Graduate Texts in Mathematics 173, Springer, Heidelberg, 2010).
doi:10.1007/978-3-642-14279-6

[11] J.H. Hattingh, Relatiewe semantiese afleibaarheid [Relative semantical entailment] (Master's dissertation (in Afrikaans), Rand Afrikaans University, Johannesburg, 1985).

[12] T.W. Haynes, S. Hedetniemi and P. Slater, Fundamentals of Domination in Graphs (Marcel Dekker Inc., New York, 1998).

[13] P. Hell and J. Nešetřil, Graphs and Homomorphisms (Oxford University Press, Oxford, 2004).

[14] W. Hodges, First-order Model Theory (The Stanford Encyclopedia of Philosophy (Summer 2009 Edition), E.N. Zalta (Ed.), 2009).
http://plato.stanford.edu/archives/sum2009/entries/modeltheory-fol

[15] R. Kellerman, private communication.

[16] A. Robinson, Introduction to Model Theory and to the Metamathematics of Algebra (North-Holland, Amsterdam, 1963).

[17] A. Salomaa, *On color-families of graphs*, Ann. Acad. Sci. Fenn. Math. **6** (1981) 135–148.
doi:10.5186/aasfm.1981.0619

[18] E. Welzl, *Color-families are dense*, Theoret. Comput. Sci. **17** (1982) 29–41.
doi:10.1016/0304-3975(82)90129-3

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