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# $\gamma$-CYCLES IN ARC-COLORED DIGRAPHS

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**Abstract**

We call a digraph $D$ an $m$-colored digraph if the arcs of $D$ are colored with $m$ colors. A directed path (or a directed cycle) is called monochromatic if all of its arcs are colored alike. A subdigraph $H$ in $D$ is called rainbow if all of its arcs have different colors. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths of $D$ if it satisfies the two following conditions:

(i) for every pair of different vertices $u, v \in N$ there is no monochromatic path in $D$ between them, and

(ii) for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic path in $D$.

A $\gamma$-cycle in $D$ is a sequence of different vertices $\gamma = (u_0, u_1, \ldots, u_n, u_0)$ such that for every $i \in \{0, 1, \ldots, n\}$:

(i) there is a $u_iu_{i+1}$-monochromatic path, and

(ii) there is no $u_{i+1}u_i$-monochromatic path.

The addition over the indices of the vertices of $\gamma$ is taken modulo $(n+1)$. If $D$ is an $m$-colored digraph, then the closure of $D$, denoted by $\mathfrak{C}(D)$, is the $m$-colored multidigraph defined as follows: $V(\mathfrak{C}(D)) = V(D)$, $A(\mathfrak{C}(D)) =$

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$A(D) \cup \{(u,v)$ with color $i \mid$ there exists a $uv$-monochromatic path colored $i$ contained in $D\}$.

In this work, we prove the following result. Let $D$ be a finite $m$-colored digraph which satisfies that there is a partition $C = C_1 \cup C_2$ of the set of colors of $D$ such that:

(1) $D[\widehat{C}_i]$ (the subdigraph spanned by the arcs with colors in $C_i$) contains no $\gamma$-cycles for $i \in \{1,2\}$;

(2) If $\mathfrak{C}(D)$ contains a rainbow $C_3 = (x_0, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then $(x_0, w) \in A(\mathfrak{C}(D))$ or $(z, x_0) \in A(\mathfrak{C}(D))$;

(3) If $\mathfrak{C}(D)$ contains a rainbow $P_3 = (u, z, w, x_0)$ involving colors of $C_1$ and $C_2$, then at least one of the following pairs of vertices is an arc in $\mathfrak{C}(D)$: $(u,w)$, $(w,u)$, $(x_0,u)$, $(u,x_0)$, $(x_0,w)$, $(z,u)$, $(z,x_0)$.

Then $D$ has a kernel by monochromatic paths.

This theorem can be applied to all those digraphs that contain no $\gamma$-cycles. Generalizations of many previous results are obtained as a direct consequence of this theorem.

**Keywords:** digraph, kernel, kernel by monochromatic paths, $\gamma$-cycle.

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