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DECOMPOSABILITY OF ABSTRACT AND PATH-INDUCED CONVEXITIES IN HYPERGRAPHS

Francesco Mario Malvestuto

and

Marina Moscarini

Department of Informatics
Sapienza University of Rome
Via Salaria 113, 00198 Roma, Italy

e-mail: malvestuto@di.uniroma1.it
moscarini@di.uniroma1.it

Abstract

An *abstract convexity space* on a connected hypergraph H with vertex set $V(H)$ is a family C of subsets of $V(H)$ (to be called the *convex sets* of H) such that: (i) C contains the empty set and $V(H)$, (ii) C is closed under intersection, and (iii) every set in C is connected in H. A convex set X of H is a *minimal vertex convex separator* of H if there exist two vertices of H that are separated by X and are not separated by any convex set that is a proper subset of X. A nonempty subset X of $V(H)$ is a *cluster* of H if in H every two vertices in X are not separated by any convex set. The *cluster hypergraph* of H is the hypergraph with vertex set $V(H)$ whose edges are the maximal clusters of H. A convexity space on H is called *decomposable* if it satisfies the following three properties:

- (C1) the cluster hypergraph of H is acyclic,
- (C2) every edge of the cluster hypergraph of H is convex,
- (C3) for every nonempty proper subset X of $V(H)$, a vertex v does not belong to the convex hull of X if and only if v is separated from X in H by a convex cluster.

It is known that the *monophonic convexity* (i.e., the convexity induced by the set of chordless paths) on a connected hypergraph is decomposable.

In this paper we first provide two characterizations of decomposable convexities and then, after introducing the notion of a *hereditary path family* in a connected hypergraph H, we show that the convexity space on H induced by any hereditary path family containing all chordless paths (such as the families of simple paths and of all paths) is decomposable.

Keywords: convex hull, hypergraph convexity, path-induced convexity, convex geometry.

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