Discussiones Mathematicae
Graph Theory 34 (2014) 467–495
doi:10.7151/dmgt.1758

INDUCED ACYCLIC TOURNAMENTS IN RANDOM DIGRAPHS: SHARP CONCENTRATION, THRESHOLDS AND ALGORITHMS[1]

Kunal Dutta and C.R. Subramanian

The Institute of Mathematical Sciences
Taramani, Chennai–600113, India

e-mail: kdutta@imsc.res.in
crs@imsc.res.in

Abstract

Given a simple directed graph $D = (V, A)$, let the size of the largest induced acyclic tournament be denoted by $mat(D)$. Let $D \in \mathcal{D}(n,p)$ (with $p = p(n)$) be a *random* instance, obtained by randomly orienting each edge of a random graph drawn from $\mathcal{G}(n, 2p)$. We show that $mat(D)$ is asymptotically almost surely (a.a.s.) one of only 2 possible values, namely either b^* or $b^* + 1$, where $b^* = \lfloor 2(\log_r n) + 0.5 \rfloor$ and $r = p^{-1}$.

It is also shown that if, asymptotically, $2(\log_r n) + 1$ is not within a distance of $w(n)/(\ln n)$ (for any sufficiently slow $w(n) \to \infty$) from an integer, then $mat(D)$ is $\lfloor 2(\log_r n) + 1 \rfloor$ a.a.s. As a consequence, it is shown that $mat(D)$ is 1-point concentrated for all n belonging to a subset of positive integers of density 1 if p is independent of n. It is also shown that there are functions $p = p(n)$ for which $mat(D)$ is provably *not* concentrated in a single value. We also establish thresholds (on p) for the existence of induced acyclic tournaments of size i which are sharp for $i = i(n) \to \infty$.

We also analyze a polynomial time heuristic and show that it produces a solution whose size is at least $\log_r n + \Theta(\sqrt{\log_r n})$. Our results are valid as long as $p \geq 1/n$. All of these results also carry over (with some slight changes) to a related model which allows 2-cycles.

Keywords: random digraphs, tournaments, concentration, thresholds, algorithms.

2010 Mathematics Subject Classification: 05C80.

References

[1] A preliminary version of parts of this work appeared as an extended abstract in LATIN, 2010, Oaxaca, Mexico.

[1] D. Achlioptas and A. Naor, *The two possible values of the chromatic number of a random graph*, Ann. of Math. **162** (2005) 1333–1349.
doi:10.4007/annals.2005.162.1335

[2] N. Alon and M. Krivelevich, *The concentration of the chromatic number of random graphs*, Combinatorica **17** (1997) 303–313.
doi:10.1007/BF01215914

[3] N. Alon and J.H. Spencer, The Probabilistic Method (Wiley International, 2001).

[4] B. Bollobás, Random Graphs (2nd Edition, Cambdrige University Press, 2001).
doi:10.1017/CBO9780511814068

[5] B. Bollobás and P. Erdős, *Cliques in random graphs*, Math. Proc. Cambridge Philos. Soc. **80** (1988) 419–427.
doi:10.1017/S0305004100053056

[6] B.K. Rosen, *Robust linear algorithms for cutsets*, J. Algorithms **3** (1982) 205–217.
doi:10.1016/0196-6774(82)90020-7

[7] K. Dutta and C.R. Subramanian, *Largest induced acyclic tournament in random digraphs*: *A* 2-*point concentration*, in: Proceedings of LATIN-2010 (9th Latin American Theoretical Informatics Symposium), Oaxaca, Mexico, April 2010.

[8] K. Dutta and C.R. Subramanian, *Induced acyclic subgraphs in random digraphs*: *Improved bounds*, in: Proceedings of AofA-2010 (21st Internatioanl Meeting on Probabilistic and Asymptotic Methods for the Analysis of Algorithms), Vienna, Austria, June 2010, to appear.

[9] R.W. Floyd *Assigning meaning to programs*, Proc. Sympos. Appl. Math. **19** (1967) 19–32.

[10] A.M. Frieze, *On the independence number of random graphs*, Discrete Math. **81** (1990) 171–176.
doi:10.1016/0012-365X(90)90149-C

[11] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide To The Theory of NP-Completeness (W.H.Freeman, San Francisco, 1978).

[12] J. Hastad, *Clique is hard to approximate within* $n^{1-\varepsilon}$, Acta Math. **182** (1999) 105–142.
doi:10.1007/BF02392825

[13] S. Janson, T. Łuczak and A. Ruciński, Random Graphs (John Wiley & Sons, Inc., 2000).
doi:10.1002/9781118032718

[14] S. Khot, *Improved inapproximability results for maxclique, chromatic number and approximate graph coloring*, in: Proceedings of the 42nd IEEE Symp. Foundations of Computer Science (FOCS 2001) 600–609.

[15] M. Krivelevich and B. Sudakov, *Coloring random graph*, Inform. Process. Lett. **67** (1998) 71–74.
doi:10.1016/S0020-0190(98)00092-1

[16] T. Łuczak, *A note on the sharp concentration of the chromatic number of random graphs*, Combinatorica **11** (1991) 295–297.
doi:10.1007/BF01205080

[17] C. Lund and M. Yannakakis, *The approximation of maximum subgraph problems*, Proceedings of the 20th International Colloquium on Automata, Languages and Programming (ICALP'93), Lecture Notes in Comput. Sci. **700** (1993) 40–51.

[18] M. Cai, X. Deng and W. Zang, *An approximation algorithm for feedback vertex set in tournaments* SIAM J. Comput. **30** (2001) 1993–2007.
doi:10.1137/S0097539798338163

[19] R. Motwani and P. Raghavan, Randomized Algorithms (Cambridge University Press, 1995).
doi:10.1017/CBO9780511814075

[20] C.H. Papadimitriou and M. Yannakakis *Optimization, approximation, and complexity classes*, J. Comput. System Sci. (Special issue for the 20th ACM Symposium on Theory of Computing) **43** (1991) 425–440.

[21] E. Speckenmeyer, *On feedback problems in digraphs*, Proceedings of the 15th International Workshop on Graph Theoretic Concepts in Computer Science (WG'89), Springer-Verlag, Lecture Notes in Comput. Sci. **411** (1990) 218–231.
doi:10.1007/3-540-52292-1_16

[22] J.H. Spencer and C.R. Subramanian, *On the size of induced acyclic subgraphs in random digraphs*, Discrete Math. Theor. Comput. Sci. **10** (2008) 47–54.

[23] C.R. Subramanian, *Finding induced acyclic subgraphs in random digraphs*, Electron. J. Combin. **10** (2003) #R46.

1. Appendix

1.1. $mat(D)$ versus $\omega(G)$

The following lemma relates the probabilities in the two models $\mathcal{D}(n,p)$ and $\mathcal{G}(n,p)$ for having, respectively, tournaments and cliques of specific sizes. Its proof is similar to the proof of an analogous relationship involving $mas(D)$ and $\alpha(G)$ (maximum size of an independent set in G) established in [23].

Lemma 1.1. *For any positive integer b, for a random digraph $D \in \mathcal{D}(n,p)$,*

$$\mathbf{Pr}[mat(D) \geq b] \geq \mathbf{Pr}[\omega(G) \geq b],$$

where $G \in \mathcal{G}(n,p)$.

Proof. Given a linear ordering σ of vertices of D and a subset A of size b, we say that $D[A]$ is consistent with σ if for every $\sigma_i, \sigma_j \in A$ with $i < j$, $D[A]$ has the arc (σ_i, σ_j).

Let τ denote an arbitrary but fixed ordering of V. Once we fix τ, the spanning subgraph of D formed by arcs of the form $(\tau(i), \tau(j))$ $(i < j)$ is having the same distribution as $\mathcal{G}(n,p)$. Hence, for any A, the event of $D[A]$ being consistent with τ is equivalent to the event of A inducing a clique in $\mathcal{G}(n,p)$. Hence,

$$\begin{aligned}
\mathbf{Pr}(\ mat(D) \geq b) &= \mathbf{Pr}(\ \exists A,\ |A| = b,\ D[A] \text{ is an acyclic tournament}) \\
&= \mathbf{Pr}(\ \exists A,\ |A| = b,\ \exists \sigma,\ D[A] \text{ is consistent with } \sigma) \\
&= \mathbf{Pr}(\ \exists \sigma,\ \exists A,\ |A| = b,\ D[A] \text{ is consistent with } \sigma) \\
&\geq \mathbf{Pr}(\ \exists A,\ |A| = b,\ D[A] \text{ is consistent with } \tau\ ) \\
&= \mathbf{Pr}(\ \omega(G) \geq b).
\end{aligned}$$

Hence it is natural that we have a bigger upper bound for $mat(D)$ than we have for $\omega(G)$. ■

Note: Recall that we first draw an undirected $G \in \mathcal{G}(n, 2p)$ and then choose uniformly randomly an orientation of $E(G)$. Hence, for any fixed $A \subseteq V$ of size b with $b = \omega(1)$,

$$\mathbf{Pr}(\ D[A] \text{ is an acyclic tournament } \mid G[A] \text{ induces a clique }) = \frac{b!}{2^{\binom{b}{2}}} = o(1).$$

However, there are so many cliques of size b in G that one of them manages to induce an acyclic tournament.

1.2. Proof of Theorem ??

We reduce the NP-complete Maximum Clique problem MC(G, k) to the MAT$(D$, $k)$ problem as follows. Given an instance $(G = (V, E), k)$ of the first problem, compute an instance $f(G) = (G' = (V, A), k)$ in polynomial time where

$$A = \{(u, v) : uv \in E, u < v\}.$$

Clearly, G' is a dag and it is easy to see that a set $V' \subseteq V$ induces a clique in G if and only if V' induces an acyclic tournament in G'. This establishes that MAT(D, k) is NP-hard even if D is restricted to be a dag.

The inapproximability of MAT(D) follows from the following observation. Note that the reduction $G \to f(G)$ is an L-reduction in the sense of [20], since $|f(G)| = |G|$ and $\omega(G) = mat(G')$. Hence, any inapproximability result on maximum clique in undirected graphs (for example [12, 14]), implies a similar inapproximability for the MAT(D) problem.

1.3. Proof of Claim ??

Order the vertices of U along a Hamilton path P (if any exists) of H. An arc $(u, v) \in A$ is a forward arc if u comes before v in P and is a backward arc otherwise. Since H is acyclic, any arc $(v, u) \in A$ must be a forward arc, since otherwise the segment of P from u to v along with (v, u) forms a cycle in H.

Now if there is another Hamilton path Q in H, $Q \neq P$, then walking along P, consider the first vertex a where Q differs from P. Then in the path Q, a is visited immediately after some vertex a' that comes after a in P. But this implies that (a', a) is a backward arc in H contradicting the observation earlier that H has no backward arc.

1.4. Remaining cases of Theorem ??

For $1/wn \leq p < 1/n$,

$$E[X(n,4)] = \binom{n}{4} \cdot 4! \cdot p^{\binom{4}{2}} \leq n^4 p^6 \ \leq \ (1/n^2) = o(1).$$

Now, an acyclic tournament of size 2 is simply an edge which a.a.s. exists since:

$$\mathbf{Pr}[mat(D) < 2] = \mathbf{Pr}[D \text{ is the empty graph}] = (1-2p)^{\binom{n}{2}} \leq e^{-n(n-1)p} = o(1),$$

since $p \geq 1/wn \geq w/n^2$. Hence, when $1/wn \leq p \leq 1/n$, $mat(D) \in \{2, 3\}$, a.a.s.

For $wn^{-2} \leq p < 1/wn$,

$$E[X(n,3)] = \binom{n}{3} \cdot 3! \cdot p^{\binom{3}{2}} \leq n^3 p^3 = o(1) \text{ since } np = o(1).$$

The proof for $mat(D) \geq 2$ is the same as in the previous case, since $n^2 p = \omega(1)$, and hence, at least one arc will exist, a.a.s. So when $w/n^2 \leq p \leq 1/wn$, $mat(D) = 2$, a.a.s.

For $(wn^2)^{-1} \leq p \leq w/n^2$, $E[X(n,3)] = o(1)$, as in the previous case, and so $mat(D) = 1$ or 2, a.a.s. When $p < (wn^2)^{-1}$, $mat(D) = 1$ since D a.a.s. has no directed edge.

Received 25 April 2011
Revised 29 April 2013
Accepted 29 April 2013